Measures of Potential Output from an Estimated DSGE Model of the United States

TECHNICAL APPENDIX

(Not for Publication)

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1 INTRODUCTORY NOTES

- Model features unit root shocks to labor-augmenting technology S_t^y and to the inflation target π_t^*.
- To make the model stationary, this requires the following:
  - Scaling of real variables, say x_t: $\check{x}_t = x_t / S_t^y$.
  - Scaling of nominal variables, say Z_t: $z_t = Z_t / P_t^*$ ($P_t^* =$ target price path).
- Rescaling is done throughout this Appendix. We therefore present the unit root processes first.
- Technology Shocks:
  - In Levels:

$$S_t^y = S_{t-1}^y g_t$$

$$g_t = g_t^{gr} g_t^{iid}$$

$$\ln g_t^{gr} = (1 - \rho_g) \ln \bar{g} + \rho_g \ln g_{t-1}^{gr} + \hat{\varepsilon}_t^{gr}$$

$$\ln g_t^{iid} = \hat{\varepsilon}_t^{iid}$$

  - Linearized:

$$\hat{g}_t = \hat{g}_t^{gr} + \hat{g}_t^{iid} \tag{1}$$

$$\hat{g}_t^{gr} = \rho_g \hat{g}_{t-1}^{gr} + \hat{\varepsilon}_t^{gr} \tag{2}$$

$$\hat{g}_t^{iid} = \hat{\varepsilon}_t^{iid} \tag{3}$$

- Inflation Target Shocks:
  - In Levels:

$$\pi_t^* = \pi_{t-1}^* \varepsilon_t^{\pi^*}$$

  - Linearized:

$$\hat{\pi}_t^* = \hat{\pi}_{t-1}^* + \hat{\varepsilon}_t^{\pi^*} \tag{4}$$

- Linearization of any variable $\check{x}_t$ around its steady state $\bar{x}$: $\hat{x}_t = (\check{x}_t - \bar{x})/\bar{x}$.
- All inflation and interest rates are gross rates.
- Indices for different heterogeneous agents:
  - i for households.
  - j for manufacturing firms.
  - z for financial intermediaries.

2 HOUSEHOLDS

2.1 Optimization Problem

- Objective Function for Household i:

$$Max \quad E_0 \sum_{t=0}^{\infty} \beta^t \left\{ S_t^c (1 - \frac{v}{\bar{g}}) \log(H_t(i)) - S_t^L \psi \frac{L_t(i)^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} + \frac{a}{1-\epsilon} \left(\frac{M_t(i)}{P_t} \right)^{1-\epsilon} \right\} \text{ , where}$$

$$H_t(i) = C_t(i) - \nu C_{t-1} \text{ (external habit)}$$

$$C_t(i) = \left(\int_0^1 c_t(i,j)^{\frac{\sigma_t^p - 1}{\sigma_t^p}} dj \right)^{\frac{\sigma_t^p}{\sigma_t^p - 1}} \quad , \quad P_t = \left(\int_0^1 P_t(j)^{1-\sigma_t^p} dj \right)^{\frac{1}{1-\sigma_t^p}}$$

- Budget Constraint (multiplier = $\lambda_t(i)/P_t$):

$$\begin{aligned} B_t(i) + B_t^D(i) &= i_{t-1} \left(B_{t-1}(i) + B_{t-1}^D(i) \right) + M_{t-1}(i) - M_t(i) \\ &\quad + W_t(i) L_t(i) + \int_0^1 \Pi_t(i,j) dj + \int_0^1 \Pi_t(i,z) dz - P_t \tau_t(i) \\ &\quad - P_t C_t(i) - W_t \frac{\phi_w}{2} \frac{(L_t(i) - \ell_t)^2}{\ell_t} \end{aligned}$$

- Cost of deviating from "normal" labor supply of other households:
  - Quadratic in the % deviation from "normal" labor supply $\frac{\phi_w}{2} (L_t(i) - \ell_t)^2 / \ell_t^2$.
  - Proportional to the aggregate wage bill $W_t \ell_t$.
- Assume complete contingent claims markets for labor income and identical initial endowments of capital, bonds and money. Then all FOC are the same except for labor supply. Therefore drop subscript i except in the wage setting problem. The latter is dealt with later, after firm and intermediary price setting.

2.2 First-Order Conditions

- FOC for B_t:

$$\lambda_t = \beta i_t E_t \left(\frac{\lambda_{t+1}}{\pi_{t+1}^p} \right)$$

  - Rescaled by technology ($\check{\lambda}_t = \lambda_t S_t^Y$) and by the inflation target ($\check{i}_t = i_t / \pi_t^*$, $\check{\pi}_{t+1}^p = \pi_{t+1}^p / \pi_{t+1}^*$):

$$\check{\lambda}_t = \beta \check{i}_t E_t \left(\frac{\check{\lambda}_{t+1}}{\check{\pi}_{t+1}^p g_{t+1} \varepsilon_{t+1}^{\pi^*}} \right)$$

– Steady state ($\bar{r} = \bar{\imath}/\pi^*$):

$$\bar{r} = \bar{g}/\beta$$

– Linearization (remember that $E_t \hat{\varepsilon}^{\pi^*}_{t+1} = 0$):

$$\hat{\lambda}_t = \hat{\imath}_t + E_t \left(\hat{\lambda}_{t+1} - \hat{\pi}^p_{t+1} - \hat{g}_{t+1} \right) \tag{5}$$

- FOC for C_t:

$$\frac{S^c_t(1 - \frac{v}{\bar{g}})}{H_t} = \lambda_t$$

– Rescaled by technology ($\check{H}_t = H_t/S^y_t$):

$$\frac{S^c_t(1 - \frac{v}{\bar{g}})}{\check{H}_t} = \check{\lambda}_t$$

– Steady state:

$$\frac{(1 - \frac{v}{\bar{g}})}{\bar{H}} = \bar{\lambda}$$

– Linearization:

$$\hat{S}^c_t - \hat{H}_t = \hat{\lambda}_t \tag{6}$$

- Habit H_t:

$$H_t = C_t - \nu C_{t-1}$$

– Rescaled by technology ($\check{C}_t = C_t/S^y_t$):

$$\check{H}_t = \check{C}_t - \nu \frac{\check{C}_{t-1}}{g_t}$$

– Steady state:

$$\bar{H} = \bar{C}(1 - \frac{\nu}{\bar{g}})$$

– Linearization:

$$\hat{H}_t = \frac{1}{1 - \frac{\nu}{\bar{g}}} \hat{C}_t - \frac{\frac{\nu}{\bar{g}}}{1 - \frac{\nu}{\bar{g}}} \left(\hat{C}_{t-1} - \hat{g}_t \right) \tag{7}$$

- Wage Setting: See below, after derivation of price setting.

3 CAPITAL GOODS PRODUCERS

- Investment Decision:
  - Investment adjustment costs:

$$G_{I,t} = \frac{\phi_I}{2} I_t \left(\frac{(I_t/g_t) - I_{t-1}}{I_{t-1}} \right)^2$$

  - Optimization problem:

$$\underset{\{I_{t+s}\}_{s=0}^{\infty}}{Max} E_t \Sigma_{s=0}^{\infty} \beta^s \frac{\lambda_{t+s}}{\lambda_t} \left[q_{t+s} \left(\tilde{K}_{t+s} + I_{t+s} \right) - q_{t+s} \tilde{K}_{t+s} - S_{t+s}^{inv} I_{t+s} - G_{I,t+s} \right]$$

  - FOC:

$$q_t = S_t^{inv} + \phi_I \left(\frac{\check{I}_t}{\check{I}_{t-1}} \right) \left(\frac{\check{I}_t - \check{I}_{t-1}}{\check{I}_{t-1}} \right) - E_t \beta \frac{\check{\lambda}_{t+1}}{\check{\lambda}_t} \phi_I \left(\frac{\check{I}_{t+1}}{\check{I}_t} \right)^2 \left(\frac{\check{I}_{t+1} - \check{I}_t}{\check{I}_t} \right)$$

  - Steady state:

$$\bar{q} = 1$$

  - Linearization:

$$\hat{q}_t = \phi_I \left(\hat{I}_t - \hat{I}_{t-1} \right) - \beta \phi_I \left(\hat{I}_{t+1} - \hat{I}_t \right) + \hat{S}_t^{inv} \tag{8}$$

- Capital Accumulation:

$$K_t = (1 - \Delta) K_{t-1} + I_t$$

  - Rescaled by technology:

$$\check{K}_t = (1 - \Delta) \frac{\check{K}_{t-1}}{g_t} + \check{I}_t$$

  - Steady state:

$$\bar{I} = \frac{(\bar{g} + \Delta - 1)}{\bar{g}} \bar{K}$$

  - Linearization:

$$\bar{K} \hat{K}_t = (1 - \Delta) \frac{\bar{K}}{\bar{g}} \left(\hat{K}_{t-1} - \hat{g}_t \right) + \bar{I} \hat{I}_{t-1} \tag{9}$$

4 ENTREPRENEURS

4.1 Capacity Utilization Decision

- Optimization Problem:

$$\underset{u_t}{Max} \quad [u_t r_{k,t} - a(u_t)]\,\omega_t K_{t-1}(j)$$

- Cost function:

$$a(u_t) = \frac{1}{2}\phi_a \sigma_a (u_t)^2 + \phi_a (1-\sigma_a) u_t + \phi_a \left(\frac{\sigma_a}{2} - 1\right)$$

- FOC:

$$r_{k,t} = \phi_a \sigma_a u_t + \phi_a (1-\sigma_a)$$

- Steady state ($\bar{u} = 1$):

$$\bar{r}_k = \phi_a$$

- Linearization:

$$\hat{r}_{k,t} = \sigma_a \hat{u}_t \tag{10}$$

4.2 Real Return to Utilized Capital

- Definition (nominal return $Ret_{k,t} = E_t\left(ret_{k,t}\pi^p_{t+1}\right)$):

$$ret_{k,t} = E_t \frac{u_{t+1} r_{k,t+1} - a(u_{t+1}) + (1-\Delta)\, q_{t+1}}{q_t}$$

- Steady state:

$$\overline{ret}_k = 1 - \Delta + \bar{r}_k$$

- Linearization:

$$\widehat{ret}_{k,t} = \frac{1-\Delta}{1-\Delta+\bar{r}_k} E_t \hat{q}_{t+1} - \hat{q}_t + \frac{\bar{r}_k}{1-\Delta+\bar{r}_k} E_t \hat{r}_{k,t+1} \tag{11}$$

4.3 Bank's Zero Profit or Participation Constraint

- Original Constraint:

$$i_t B_t^D(j) = (1 - F(\bar{\omega}_{t+1}))\, i_{B,t+1} B_t^D(j) + \left(1 - \xi_{t+1}\right) \int_0^{\bar{\omega}_{t+1}} Q_t K_t(j) Ret_{k,t} \omega f(\omega) d\omega$$

  - $i_t =$ non-contingent interest rate paid by bank to depositors.
  - $B_t^D(j) =$ nominal loan amount.
  - $\bar{\omega}_{t+1} =$ cutoff level for next period productivity, below = bankruptcy, above = pay loan in full. ($\bar{\omega}_{t+1}$ is a function of $t+1$ shocks.)
  - $i_{B,t+1} =$ gross nominal rate of interest to be paid in full if ω_{t+1} is high enough. ($i_{B,t+1}$ is a function of $t+1$ shocks.)
  - $F(.) =$ cdf of the log-normal distribution of ω, with $E(\omega_t) = 1$ and $Var(\omega_t) = \sigma_t^2$.
  - $(1 - \xi_{t+1}) =$ fraction of project value that the bank can recover in bankruptcy.
  - Cutoff $\bar{\omega}_{t+1}$ condition

$$Ret_{k,t} \bar{\omega}_{t+1} Q_t K_t(j) = i_{B,t+1} B_t^D(j)$$

  - Balance sheet constraint ($N_t(j) =$ nominal net worth):

$$Q_t K_t(j) = N_t(j) + B_t^D(j)$$

  - Rewritten participation constraint using the foregoing:

$$\begin{aligned} & \left[(1 - F(\bar{\omega}_{t+1}))\, \bar{\omega}_{t+1} + \left(1 - \xi_{t+1}\right) \int_0^{\bar{\omega}_{t+1}} \omega f(\omega) d\omega \right] Ret_{k,t} Q_t K_t(j) \\ = \;& i_t Q_t K_t(j) - i_t N_t(j) \end{aligned}$$

- Definitions:
  - Nominal capital earnings: $Ret_{k,t} q_t K_t(j)$
  - Lender's gross share in capital earnings:

$$\Gamma(\bar{\omega}_{t+1}) \equiv \int_0^{\bar{\omega}_{t+1}} \omega_{t+1} f(\omega_{t+1}) d\omega_{t+1} + \bar{\omega}_{t+1} \int_{\bar{\omega}_{t+1}}^{\infty} f(\omega_{t+1}) d\omega_{t+1}$$

  - Lender's monitoring costs share in capital earnings:

$$\xi_{t+1} G(\bar{\omega}_{t+1}) = \xi_{t+1} \int_0^{\bar{\omega}_{t+1}} \omega_{t+1} f(\omega_{t+1}) d\omega_{t+1}$$

  - Entrepreneur's share in capital earnings

$$1 - \Gamma(\bar{\omega}_{t+1}) = \int_{\bar{\omega}_{t+1}}^{\infty} (\omega_{t+1} - \bar{\omega}_{t+1})\, f(\omega_{t+1}) d\omega_{t+1}$$

- Aggregated Participation Constraint (using the foregoing definitions):

$$\frac{q_{t-1}\check{K}_{t-1}}{\check{n}_{t-1}}\frac{r\check{e}t_{k,t}^{m1}}{\check{r}_t^{m1}}\left(\Gamma(\bar{\omega}_t)-\xi_t G(\bar{\omega}_t)\right)-\frac{q_{t-1}\check{K}_{t-1}}{\check{n}_{t-1}}+1=0$$

  - Note the time subscripts: This has to hold in each state of the world = ex-post.
  - Trick to make our timing conventions work:

$$\begin{aligned} r\check{e}t_{k,t}^{m1} &= r\check{e}t_{k,t-1} \\ \check{r}_t^{m1} &= \check{r}_{t-1} \end{aligned}$$

  - Steady state:

$$\bar{n}=\bar{K}\left(1-\frac{\overline{ret}_k}{\bar{r}}\left(\bar{\Gamma}-\bar{\xi}\bar{G}\right)\right)$$

  - Linearization (superscripts ω and σ indicate partial derivatives, in steady state, with respect to these two variables):

$$\begin{aligned} 0 &= \left(\frac{\bar{K}}{\bar{n}}-1\right)\left(\widehat{ret}_{k,t}^{m1}-\hat{r}_t^{m1}\right)-\left(\hat{q}_{t-1}+\hat{K}_{t-1}-\hat{n}_{t-1}\right) \\ &+\left(\frac{\bar{K}}{\bar{n}}-1\right)\frac{\bar{\Gamma}^{\omega}-\bar{\xi}\bar{G}^{\omega}}{\bar{\Gamma}-\bar{\xi}\bar{G}}\overline{\omega}\widehat{\overline{\omega}}_t+\left(\frac{\bar{K}}{\bar{n}}-1\right)\frac{\bar{\Gamma}^{\sigma}-\bar{\xi}\bar{G}^{\sigma}}{\bar{\Gamma}-\bar{\xi}\bar{G}}\bar{\sigma}\hat{\sigma}_t \end{aligned} \tag{12}$$

4.4 Entrepreneur's Optimal Loan Contract

- Entrepreneur's Optimization Problem ($\tilde{\lambda}_t$ = multiplier of the participation constraint):

$$\underset{K_t(j),\bar{\omega}_{t+1}}{Max}\; E_t\left\{\left(1-\Gamma(\bar{\omega}_{t+1})\right)Ret_{k,t}Q_tK_t(j)\right.$$
$$\left.+\tilde{\lambda}_t\left[\left(\Gamma(\bar{\omega}_{t+1})-\xi_{t+1}G(\bar{\omega}_{t+1})\right)Ret_{k,t}Q_tK_t(j)-i_tQ_tK_t(j)+i_tN_t(j)\right]\right\}$$

  - Note the expectations operator: Entrepreneur is risk-neutral and absorbs all aggregate risk.
  - Divide through by $i_tN_t(j)$, normalize, and replace nominal by real returns:

$$\begin{aligned} &\underset{K_t(j),\bar{\omega}_{t+1}}{Max}\left\{\left(1-\Gamma(\bar{\omega}_{t+1})\right)\frac{r\check{e}t_{k,t}}{\check{r}_t}\frac{q_t\check{K}_t(j)}{\check{n}_t(j)}\right. \\ &\left.+\tilde{\lambda}_t\left[\left(\Gamma(\bar{\omega}_{t+1})-\xi_{t+1}G(\bar{\omega}_{t+1})\right)\frac{r\check{e}t_{k,t}}{\check{r}_t}\frac{q_t\check{K}_t(j)}{\check{n}_t(j)}-\frac{q_t\check{K}_t(j)}{\check{n}_t(j)}+1\right]\right\} \end{aligned}$$

  - FOC with respect to $\bar{\omega}_{t+1}$:

$$-\Gamma_{t+1}^{\omega}\frac{r\check{e}t_{k,t}}{\check{r}_t}\frac{q_t\check{K}_t(j)}{\check{n}_t(j)}+\tilde{\lambda}_t\left\{\left(\Gamma_{t+1}^{\omega}-\xi_{t+1}G_{t+1}^{\omega}\right)\frac{r\check{e}t_{k,t}}{\check{r}_t}\frac{q_t\check{K}_t(j)}{\check{n}_t(j)}\right\}=0$$

This implies:

$$\tilde{\lambda}_t = \frac{\Gamma^\omega_{t+1}}{\Gamma^\omega_{t+1} - \xi_{t+1} G^\omega_{t+1}}$$

- Optimal loan contract (FOC with respect to $\check{K}_t(j)$):

$$E_t \left\{ (1-\Gamma_{t+1}) \frac{r\check{e}t_{k,t}}{\check{r}_t} + \frac{\Gamma^\omega_{t+1}}{\Gamma^\omega_{t+1} - \xi_{t+1} G^\omega_{t+1}} \left[\frac{r\check{e}t_{k,t}}{\check{r}_t} \left(\Gamma_{t+1} - \xi_{t+1} G_{t+1} \right) - 1 \right] \right\} = 0$$

  - Steady state:

$$\overline{ret}_k = \frac{\bar{r}\bar{\tilde{\lambda}}}{1 - \bar{\Gamma} + \bar{\tilde{\lambda}} \left(\bar{\Gamma} - \bar{\xi}\bar{G} \right)}$$

  - Linearization:

$$\begin{aligned} 0 \;=\; & \bar{\tilde{\lambda}} \left(\widehat{ret}_{k,t} - \hat{r}_t \right) - \left(1 - \bar{\Gamma} \right) \frac{\bar{r}^k}{\bar{r}} \left(\frac{\bar{\Gamma}^{\omega\omega} - \bar{\tilde{\lambda}} \left(\bar{\Gamma}^{\omega\omega} - \bar{\xi}\bar{G}^{\omega\omega} \right)}{\bar{\Gamma}^\omega} \right) \overline{\omega} E_t \widehat{\bar{\omega}}_{t+1} \\ & + \frac{\bar{r}^k}{\bar{r}} \left[-\Gamma^\sigma + \bar{\tilde{\lambda}} \left(\bar{\Gamma}^\sigma - \bar{\xi}\bar{G}^\sigma \right) - \left(1 - \bar{\Gamma} \right) \left(\frac{\bar{\Gamma}^{\omega\sigma} - \bar{\tilde{\lambda}} \left(\bar{\Gamma}^{\omega\sigma} - \bar{\xi}\bar{G}^{\omega\sigma} \right)}{\bar{\Gamma}^\omega} \right) \right] \bar{\sigma} E_t \hat{\sigma}_{t+1} \end{aligned} \tag{13}$$

4.5 Net Wealth Accumulation

- Nominal net wealth accumulation ($div_t =$ lump-sum dividends to households):

$$N_t = Ret_{k,t-1} Q_{t-1} \check{K}_{t-1} \left(1 - \xi_t G_t \right) - i_{t-1} B^D_{t-1} - P_t div_t$$

- Real net wealth accumulation (using the balance sheet identity):

$$\check{n}_t = \check{r}^{m1}_t \frac{\check{n}_{t-1}}{g_t} + \frac{q_{t-1}\check{K}_{t-1}}{g_t} \left(r\check{e}t^{m1}_{k,t} \left(1 - \xi_t G_t \right) - \check{r}^{m1}_t \right) - d\check{i}v_t$$

- Dividends:

$$d\check{i}v_t = d * \check{n}_t$$

- Steady state:

$$d = \frac{\bar{r}}{\bar{g}} - 1 + \frac{\bar{K}}{\bar{g}\bar{n}} \left[\overline{ret}_k \left(1 - \bar{\xi}\bar{G} \right) - \bar{r} \right]$$

- Linearization:

$$\begin{aligned} & \bar{n}\bar{g} \left(1 + d \right) \left(\hat{n}_t + \hat{g}_t \right) \\ = \; & \bar{r} \left(\bar{n} - \bar{K} \right) \hat{r}^{m1}_t + \bar{r}\bar{n}\hat{n}_{t-1} + \bar{K} \left(\overline{ret}_k \left(1 - \bar{\xi}\bar{G} \right) - \bar{r} \right) \left(\hat{q}_{t-1} + \hat{K}_{t-1} \right) \\ & + \bar{K}\overline{ret}_k \left(1 - \bar{\xi}\bar{G} \right) \left(\widehat{ret}^{m1}_{k,t} \right) - \bar{K}\overline{ret}_k \bar{\xi} \left(\bar{G}^\omega \overline{\omega} \widehat{\bar{\omega}}_t + \bar{G}^\sigma \bar{\sigma} \hat{\sigma}_t \right) - \bar{K}\overline{ret}_k \bar{G} \hat{\xi}_t \end{aligned}$$

4.6 Closed Forms for Γ, G and Their Partial Derivatives

4.6.1 Basic Properties of Γ and G

- Repeat expressions for Γ and G for ease of reference:

$$\Gamma(\bar{\omega}_{t+1}) \equiv \int_0^{\bar{\omega}_{t+1}} \omega_{t+1} f(\omega_{t+1}) d\omega_{t+1} + \bar{\omega}_{t+1} \int_{\bar{\omega}_{t+1}}^{\infty} f(\omega_{t+1}) d\omega_{t+1}$$

$$G(\bar{\omega}_{t+1}) = \int_0^{\bar{\omega}_{t+1}} \omega_{t+1} f(\omega_{t+1}) d\omega_{t+1}$$

- First derivatives with respect to $\bar{\omega}_{t+1}$:

$$\Gamma^{\omega}_{t+1} = 1 - F(\bar{\omega}_{t+1})$$
$$G^{\omega}_{t+1} = \bar{\omega}_{t+1} f(\bar{\omega}_{t+1})$$

4.6.2 Basic Properties of the Lognormal Distribution

The assumption is that ω_t is lognormally distributed with $E(\omega_t) = 1$ and $Var(\omega_t) = \sigma_t^2$. This implies the following:

$$\ln(\omega_t) \sim N(-\frac{1}{2}\sigma_t^2, \sigma_t^2)$$
$$f(\omega_t) = \frac{1}{\sqrt{2\pi}\omega_t \sigma_t} \exp\left\{-\frac{1}{2}\left(\frac{\ln(\omega_t) + \frac{1}{2}\sigma_t^2}{\sigma_t}\right)^2\right\}$$

4.6.3 Derivations A: The Nonlinear System

- We will change integrands at various points in order to obtain solutions that can be expressed in terms of the cumulative distribution function Φ of the standard normal distribution.
- First define terms:

$$\bar{z}_t = \frac{\ln(\bar{\omega}_t) + \frac{1}{2}\sigma_t^2}{\sigma_t} \quad , \quad y_t = \frac{\ln(\omega_t) + \frac{1}{2}\sigma_t^2}{\sigma_t}$$
$$\tilde{z}_t = \frac{\ln(\bar{\omega}_t) - \frac{1}{2}\sigma_t^2}{\sigma_t} \quad , \quad \tilde{y}_t = \frac{\ln(\omega_t) - \frac{1}{2}\sigma_t^2}{\sigma_t}$$

- Manipulating the second expression in each case gives:

$$d\omega_t = \sigma_t \exp\left\{y_t \sigma_t - \frac{1}{2}\sigma_t^2\right\} dy_t$$
$$d\omega_t = \sigma_t \exp\left\{\tilde{y}_t \sigma_t + \frac{1}{2}\sigma_t^2\right\} d\tilde{y}_t$$

- Now evaluate the expressions determining Γ and G in terms of the c.d.f. $\Phi(.)$:
  - First Component:

$$\int_{\bar{\omega}_{t+1}}^{\infty} f(\omega_{t+1})d\omega_{t+1} = \int_{\bar{\omega}_{t+1}}^{\infty} \frac{1}{\sqrt{2\pi}\omega_{t+1}\sigma_{t+1}} \exp\left\{-\frac{1}{2}\left(\frac{\ln(\omega_{t+1})+\frac{1}{2}\sigma_{t+1}^2}{\sigma_{t+1}}\right)^2\right\} d\omega_{t+1}$$

$$= \int_{\bar{z}_{t+1}}^{\infty} \frac{\sigma_{t+1}}{\sqrt{2\pi}\omega_{t+1}\sigma_{t+1}} \exp\left\{-\frac{1}{2}y_{t+1}^2\right\} \exp\left\{y_{t+1}\sigma_{t+1} - \frac{1}{2}\sigma_{t+1}^2\right\} dy_{t+1}$$

$$= \int_{\bar{z}_{t+1}}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{\omega_{t+1}} \exp\left\{-\frac{1}{2}\left(y_{t+1}^2 + \sigma_{t+1}^2 - 2y_{t+1}\sigma_{t+1}\right)\right\} dy_{t+1}$$

$$= \int_{\bar{z}_{t+1}}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{\omega_{t+1}} \exp\left\{-\frac{1}{2}\left(y_{t+1} - \sigma_{t+1}\right)^2\right\} dy_{t+1}$$

$$= \int_{\bar{z}_{t+1}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\ln(\omega_{t+1})\right\} \exp\left\{-\frac{\left(\ln(\omega_{t+1}) - \frac{1}{2}\sigma_{t+1}^2\right)^2}{2\sigma_{t+1}^2}\right\} dy_{t+1}$$

$$= \int_{\bar{z}_{t+1}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{\frac{-2\sigma_{t+1}^2\ln(\omega_{t+1}) - \left(\ln(\omega_{t+1})\right)^2 - \left(\frac{1}{2}\sigma_{t+1}^2\right)^2 + \ln(\omega_t)\sigma_{t+1}^2}{2\sigma_{t+1}^2}\right\} dy_{t+1}$$

$$= \int_{\bar{z}_{t+1}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{\left(\ln(\omega_{t+1})\right)^2 + \left(\frac{1}{2}\sigma_{t+1}^2\right)^2 + 2\ln(\omega_{t+1})\frac{1}{2}\sigma_{t+1}^2}{2\sigma_{t+1}^2}\right\} dy_{t+1}$$

$$= \int_{\bar{z}_{t+1}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{\ln(\omega_{t+1})+\frac{1}{2}\sigma_{t+1}^2}{\sigma_{t+1}}\right)^2\right\} dy_{t+1}$$

$$= \int_{\bar{z}_{t+1}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(y_{t+1}\right)^2\right\} dy_{t+1} = 1 - \Phi\left(\bar{z}_{t+1}\right)$$

  - Second Component:

$$\int_{\bar{\omega}_{t+1}}^{\infty} \omega_{t+1} f(\omega_{t+1})d\omega_{t+1} = \int_{\bar{\omega}_{t+1}}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_{t+1}} \exp\left\{-\frac{1}{2}\left(\frac{\ln(\omega_{t+1})+\frac{1}{2}\sigma_{t+1}^2}{\sigma_{t+1}}\right)^2\right\} d\omega_{t+1}$$

$$= \int_{\tilde{z}_{t+1}}^{\infty} \frac{\sigma_{t+1}}{\sqrt{2\pi}\sigma_{t+1}} \exp\left\{-\frac{1}{2}\left(\tilde{y}_{t+1} + \sigma_{t+1}\right)^2\right\} \exp\left\{\tilde{y}_{t+1}\sigma_{t+1} + \frac{1}{2}\sigma_{t+1}^2\right\} d\tilde{y}_{t+1}$$

$$= \int_{\tilde{z}_{t+1}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\tilde{y}_{t+1}^2 - \frac{1}{2}\sigma_{t+1}^2 - \tilde{y}_{t+1}\sigma_{t+1} + \tilde{y}_{t+1}\sigma_{t+1} + \frac{1}{2}\sigma_{t+1}^2\right\} d\tilde{y}_{t+1}$$

$$= \int_{\tilde{z}_{t+1}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\tilde{y}_{t+1}^2\right\} d\tilde{y}_{t+1} = 1 - \Phi\left(\tilde{z}_{t+1}\right) = 1 - \Phi\left(\bar{z}_{t+1} - \sigma_{t+1}\right)$$

– Summary:

$$\int_{\bar{\omega}_{t+1}}^{\infty} f(\omega_{t+1})d\omega_{t+1} = 1 - \Phi\left(\bar{z}_{t+1}\right)$$
$$\int_{\bar{\omega}_{t+1}}^{\infty} \omega_{t+1} f(\omega_{t+1})d\omega_{t+1} = 1 - \Phi\left(\bar{z}_{t+1} - \sigma_{t+1}\right)$$

– Final set of nonlinear equations:

$$\bar{z}_t = \frac{\ln(\bar{\omega}_t) + \frac{1}{2}\sigma_t^2}{\sigma_t}$$
$$f\left(\bar{\omega}_t\right) = \frac{1}{\sqrt{2\pi}\bar{\omega}_t\sigma_t}\exp\left\{-\frac{1}{2}\bar{z}_t^2\right\}$$
$$\Gamma_t = \Phi\left(\bar{z}_t - \sigma_t\right) + \bar{\omega}_t\left(1 - \Phi\left(\bar{z}_t\right)\right)$$
$$G_t = \Phi\left(\bar{z}_t - \sigma_t\right)$$
$$\Gamma_t^{\omega} = 1 - \Phi\left(\bar{z}_t\right)$$
$$G_t^{\omega} = \bar{\omega}_t f\left(\bar{\omega}_t\right)$$

4.6.4 Derivations B: Derivatives for the Linearized System

- Closed Form Expressions for Standard Normal cdf's:

$$\Phi\left(\bar{z}_t - \sigma_t\right) = \int_0^{\bar{z}_t - \sigma_t} \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}y_t^2\right\}dy_t$$
$$\Phi\left(\bar{z}_t\right) = \int_0^{\bar{z}_t} \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}y_t^2\right\}dy_t$$

- Auxiliary Relationships:
  – Derivatives of Γ:
    * We have the following:

$$\Gamma_t = \Phi\left(\bar{z}_t - \sigma_t\right) + \bar{\omega}_t\left(1 - \Phi\left(\bar{z}_t\right)\right)$$
$$\Gamma_t^{\omega} = 1 - \Phi\left(\bar{z}_t\right)$$

    * We take the derivative of the first expression:

$$\Gamma_t^{\omega} = \Phi'\left(\bar{z}_t - \sigma_t\right)\frac{1}{\bar{\omega}_t\sigma_t} + \left(1 - \Phi\left(\bar{z}_t\right)\right) - \bar{\omega}_t\Phi'\left(\bar{z}_t\right)\frac{1}{\bar{\omega}_t\sigma_t}$$

    * We equate this to the first expression to obtain:

$$\Phi'\left(\bar{z}_t - \sigma_t\right) = \bar{\omega}_t\Phi'\left(\bar{z}_t\right)$$

- Derivatives of G:
  - We have the following:

$$G_t = \Phi\left(\bar{z}_t - \sigma_t\right)$$
$$G_t^{\omega} = \bar{\omega}_t f\left(\bar{\omega}_t\right)$$

  - We take the derivative of the first expression:

$$G_t^{\omega} = \Phi'\left(\bar{z}_t - \sigma_t\right)\frac{1}{\bar{\omega}_t \sigma_t}$$

  - We combine this with the second expression to obtain our final two closed form relationships for derivatives of standard normal cdf's:

$$\Phi'\left(\bar{z}_t - \sigma_t\right) = \bar{\omega}_t^2 \sigma_t f\left(\bar{\omega}_t\right)$$
$$\Phi'\left(\bar{z}_t\right) = \bar{\omega}_t \sigma_t f\left(\bar{\omega}_t\right)$$

- $\Gamma_t^{\omega\omega}$:

$$\Gamma_t^{\omega\omega} = -\Phi'(\bar{z}_t)\frac{1}{\bar{\omega}_t \sigma_t} = -\frac{1}{\sqrt{2\pi}\bar{\omega}_t \sigma_t}\exp\left\{-\frac{1}{2}\bar{z}_t^2\right\} = -f\left(\bar{\omega}_t\right)$$

- $\Gamma_t^{\omega\sigma}$:

$$\begin{aligned}\Gamma_t^{\omega\sigma} &= -\Phi'(\bar{z}_t)\frac{\sigma_t^2 - \ln(\bar{\omega}_t) - \frac{1}{2}\sigma_t^2}{\sigma_t^2} \\ &= -\frac{1}{\sqrt{2\pi}\bar{\omega}_t \sigma_t}\exp\left\{-\frac{1}{2}\bar{z}_t^2\right\}\bar{\omega}_t \frac{\ln(\bar{\omega}_t) - \frac{1}{2}\sigma_t^2}{\sigma_t} \\ &= f\left(\bar{\omega}_t\right)\bar{\omega}_t\left(\bar{z}_t - \sigma_t\right)\end{aligned}$$

- Γ_t^{σ}:

$$\begin{aligned}\Gamma_t^{\sigma} &= \Phi'(\bar{z}_t - \sigma_t)\frac{-\sigma_t^2 - \ln(\bar{\omega}_t) + \frac{1}{2}\sigma_t^2}{\sigma_t^2} - \bar{\omega}_t \Phi'(\bar{z}_t)\frac{\sigma_t^2 - \ln(\bar{\omega}_t) - \frac{1}{2}\sigma_t^2}{\sigma_t^2} \\ &= \Phi'(\bar{z}_t - \sigma_t)\left(\frac{-\ln(\bar{\omega}_t) - \frac{1}{2}\sigma_t^2}{\sigma_t^2} - \frac{-\ln(\bar{\omega}_t) + \frac{1}{2}\sigma_t^2}{\sigma_t^2}\right) = -\Phi'(\bar{z}_t - \sigma_t) \\ &= -\bar{\omega}_t^2 \sigma_t f\left(\bar{\omega}_t\right)\end{aligned}$$

- $G_t^{\omega\omega}$:

$$\begin{aligned}G_t^{\omega\omega} &= f\left(\bar{\omega}_t\right) + \bar{\omega}_t\left[-\frac{1}{\sqrt{2\pi}\bar{\omega}_t \sigma_t}\frac{1}{\bar{\omega}_t}\exp\left\{-\frac{1}{2}\bar{z}_t^2\right\} + \frac{1}{\sqrt{2\pi}\bar{\omega}_t \sigma_t}(-\bar{z}_t)\exp\left\{-\frac{1}{2}\bar{z}_t^2\right\}\frac{1}{\bar{\omega}_t \sigma_t}\right] \\ &= f\left(\bar{\omega}_t\right) + \bar{\omega}_t f\left(\bar{\omega}_t\right)\left[-\frac{1}{\bar{\omega}_t} - \frac{\bar{z}_t}{\bar{\omega}_t \sigma_t}\right] = f\left(\bar{\omega}_t\right)\left[1 - 1 - \frac{\bar{z}_t}{\sigma_t}\right] \\ &= -f\left(\bar{\omega}_t\right)\frac{\bar{z}_t}{\sigma_t}\end{aligned}$$

- $G_t^{\omega\sigma}$:

$$
\begin{aligned}
G_t^{\omega\sigma} &= \bar{\omega}_t \left[-\frac{1}{\sqrt{2\pi}\bar{\omega}_t \sigma_t} \frac{1}{\sigma_t} \exp\left\{ -\frac{1}{2}\bar{z}_t^2 \right\} + \frac{1}{\sqrt{2\pi}\bar{\omega}_t \sigma_t} (-\bar{z}_t) \exp\left\{ -\frac{1}{2}\bar{z}_t^2 \right\} \frac{\sigma_t^2 - \ln(\bar{\omega}_t) - \frac{1}{2}\sigma_t^2}{\sigma_t^2} \right] \\
&= \bar{\omega}_t f\left(\bar{\omega}_t\right) \left[-\frac{1}{\sigma_t} + \frac{1}{\sigma_t} \bar{z}_t \frac{\ln(\bar{\omega}_t) - \frac{1}{2}\sigma_t^2}{\sigma_t} \right] \\
&= \frac{\bar{\omega}_t}{\sigma_t} f\left(\bar{\omega}_t\right) \left[\bar{z}_t \left(\bar{z}_t - \sigma_t \right) - 1 \right]
\end{aligned}
$$

- G_t^{σ}:

$$
\begin{aligned}
G_t^{\sigma} &= \Phi'\left(\bar{z}_t - \sigma_t\right) \frac{-\sigma_t^2 - \ln(\bar{\omega}_t) + \frac{1}{2}\sigma_t^2}{\sigma_t^2} = -\bar{\omega}_t^2 \sigma_t f\left(\bar{\omega}_t\right) \frac{\ln(\bar{\omega}_t) + \frac{1}{2}\sigma_t^2}{\sigma_t^2} \\
&= -\bar{\omega}_t^2 f\left(\bar{\omega}_t\right) \bar{z}_t
\end{aligned}
$$

4.6.5 Standard Normal cdf's and Complementary Error Functions

- General Relationship:

$$
\Phi(x) = \frac{1}{2} erfc\left(\frac{-x}{\sqrt{2}} \right)
$$

- Applied to our case:

$$
\Phi\left(\bar{z}_t - \sigma_t\right) = \frac{1}{2} erfc\left(\frac{\sigma_t - \bar{z}_t}{\sqrt{2}} \right)
$$

$$
\Phi\left(\bar{z}_t\right) = \frac{1}{2} erfc\left(\frac{-\bar{z}_t}{\sqrt{2}} \right)
$$

5 FIRMS

5.1 Cost Minimization

- Production Functions:

$$y_t(j) = \left(S_t^y \ell_t(j)\right)^{1-\alpha} k_t(j)^\alpha$$

- Real Marginal Cost:
  - In levels:

$$mc_t = \frac{A w_t^{1-\alpha} \left(r_t^k\right)^\alpha}{\left(S_t^y\right)^{1-\alpha}} \quad , \quad \text{where } A = \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}$$

  - Rescaled by technology ($\check{w}_t = w_t / S_t^y$):

$$mc_t = A \check{w}_t^{1-\alpha} \left(r_t^k\right)^\alpha$$

  - Linearized:

$$\widehat{mc}_t = (1-\alpha)\hat{w}_t + \alpha \hat{r}_t^k \tag{14}$$

- Input Demands for Aggregate Firm Sector (see next page for definitions of aggregate variables):
  - In levels:

$$\begin{aligned} \ell_t &= (1-\alpha)\frac{mc_t}{w_t}\tilde{Y}_t \\ k_t &= \alpha \frac{mc_t}{r_t^k}\tilde{Y}_t \end{aligned}$$

  - Rescaled by technology ($\check{Y}_t = \tilde{Y}_t / S_t^y$):

$$\begin{aligned} \ell_t &= (1-\alpha)\frac{mc_t}{\check{w}_t}\check{Y}_t \\ \check{k}_t &= \alpha \frac{mc_t}{r_t^k}\check{Y}_t \end{aligned}$$

  - Linearized:

$$\hat{\ell}_t = \widehat{mc}_t - \hat{w}_t + \hat{Y}_t \tag{15}$$

$$\hat{k}_t = \widehat{mc}_t - \hat{r}_t^k + \hat{Y}_t \tag{16}$$

- Relationship between physical capital K_t and utilized capital k_t:
  - In levels:
$$k_t = u_t K_{t-1}$$
  - Rescaled by technology:
$$\check{k}_t = u_t \frac{\check{K}_{t-1}}{g_t}$$
  - Steady state:
$$\bar{k} = \frac{\bar{K}}{\bar{g}}$$
  - Linearization:
$$\hat{k}_t = \hat{K}_{t-1} + \hat{u}_t - \hat{g}_t$$
- Definitions used above:
  - Labor:
    * Aggregate:
$$\ell_t = \int_0^1 \ell_t(j)dj \quad , \quad \text{where } \ell_t(j) = \left(\int_0^1 L_t(j,i)^{\frac{\sigma_t^w - 1}{\sigma_t^w}} di \right)^{\frac{\sigma_t^w}{\sigma_t^w - 1}}$$
$$L_t(i) = \int_0^1 L_t(j,i)dj$$
    * Varieties $L_t(j,i)$ supplied by households i, see above and also Section 5 on household wage setting.
    * Cost minimizing varieties demands:
$$L_t(j,i) = \ell_t(j) \left(\frac{W_t(i)}{W_t} \right)^{-\sigma_t^w} \quad , \quad \text{therefore } L_t(i) = \ell_t \left(\frac{W_t(i)}{W_t} \right)^{-\sigma_t^w}$$
  - Capital:
$$k_t = \int_0^1 k_t(j)dj$$
  - Output:
    * Aggregate:
$$\tilde{Y}_t = \int_0^1 y_t(j)dj \quad , \quad \text{while } Y_t = \left(\int_0^1 y_t(j)^{\frac{\sigma_t^p - 1}{\sigma_t^p}} dj \right)^{\frac{\sigma_t^p}{\sigma_t^p - 1}}$$
    * It is easy to show that:
$$\bar{\tilde{Y}} = \bar{Y} \quad , \quad \hat{\tilde{Y}}_t = \hat{Y}_t$$

5.2 Profit Maximization

- Discounted real profits:
  - Real revenue $\frac{P_{t+k}(j)}{P_{t+k}} y_{t+k}(j)$.
  - Real marginal cost $\frac{MC_{t+k}}{P_{t+k}} y_{t+k}(j)$.
  - Real cost of deviating from "normal" output of other firms:
    * Quadratic in the % deviation from "normal" output $\frac{\phi_p}{2} \frac{(y_{t+k}(j) - Y_{t+k})^2}{Y_{t+k}^2}$.
    * Proportional to aggregate real output Y_{t+k}.
- Pricing policy of firm j that reoptimizes at t, choosing V_t^p and v_t^p (a gross inflation rate):

$$P_{t+k}(j) = V_t^p \left(v_t^p\right)^k$$

- Profit maximization:

$$\underset{V_t^p, v_t^p}{Max} \quad E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \lambda_{t+k} \left[\frac{V_t^p \left(v_t^p\right)^k}{P_{t+k}} y_{t+k}(j) - mc_{t+k} y_{t+k}(j) - \frac{\phi_p}{2} \frac{\left(y_{t+k}(j) - Y_{t+k}\right)^2}{Y_{t+k}} \right] \quad , \ s.t.$$

$$y_{t+k}(j) = Y_{t+k} \left(\frac{V_t^p \left(v_t^p\right)^k}{P_{t+k}} \right)^{-\sigma_{t+k}^p}$$

- Substitute constraints:

$$\underset{V_t^p, v_t^p}{Max} E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \lambda_{t+k} \left[\left(\frac{V_t^p \left(v_t^p\right)^k}{P_{t+k}} \right)^{1-\sigma_{t+k}^p} Y_{t+k} - mc_{t+k} \left(\frac{V_t^p \left(v_t^p\right)^k}{P_{t+k}} \right)^{-\sigma_{t+k}^p} Y_{t+k} - \frac{\phi_p}{2} \frac{\left(y_{t+k}(j) - Y_{t+k}\right)^2}{Y_{t+k}} \right]$$

- Define terms:
  - Front-loading term:

$$p_t^p \equiv \frac{V_t^p}{P_t}$$

  - Inflation rescaled by the inflation target:

$$\breve{\pi}_t^p = \pi_t^p / \pi_t^*$$

  - Cumulative aggregate rescaled inflation:

$$\breve{\Pi}_{t,k}^p \equiv \prod_{j=1}^{k} \breve{\pi}_{t+j}^p \text{ for } k \geq 1 \ \ (\equiv 1 \text{ for } k = 0) \tag{17}$$

  - Cumulative aggregate rescaled inflation deviation:

$$\hat{\Pi}_{t,k}^p \equiv \sum_{j=1}^{k} \hat{\pi}_{t+j}^p \text{ for } k \geq 1 \ \ (\equiv 0 \text{ for } k = 0) \tag{18}$$

  - Mark-up:

$$\mu_t^p = \frac{\sigma_t^p}{\sigma_t^p - 1}$$

5.3 First-Order Conditions

- Rescaled by technology, with $\check{y}_t(j) = y_t(j)/S_t^y$.
- Rescaled by the inflation target, with $\check{v}_t^p = v_t^p/\pi_t^*$.
- FOC for V_t^p:

$$p_t^p = \frac{E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \check{\lambda}_{t+k} \check{y}_{t+k}(j) \sigma_{t+k}^p \left(mc_{t+k} + \phi_p \left(\frac{\check{y}_{t+k}(j) - \check{Y}_{t+k}}{\check{Y}_{t+k}} \right) \right)}{E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \check{\lambda}_{t+k} \check{y}_{t+k}(j) \left(\sigma_{t+k}^p - 1 \right) \left(\frac{(\check{v}_t^p)^k}{\check{\Pi}_{t,k}^p} \right)} \tag{19}$$

- FOC w.r.t. v_t^p (rescaled by technology):

$$p_t^p = \frac{E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k k \check{\lambda}_{t+k} \check{y}_{t+k}(j) \sigma_{t+k}^p \left(mc_{t+k} + \phi_p \left(\frac{\check{y}_{t+k}(j) - \check{Y}_{t+k}}{\check{Y}_{t+k}} \right) \right)}{E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k k \check{\lambda}_{t+k} \check{y}_{t+k}(j) \left(\sigma_{t+k}^p - 1 \right) \left(\frac{(\check{v}_t^p)^k}{\check{\Pi}_{t,k}^p} \right)} \tag{20}$$

- Rescaling by the inflation target - comments:
  - This takes the form of dividing all price levels by the target price level P_t^*.
  - In the preceding formulas this only affects $E_t \frac{(v_t^p)^k}{\Pi_{t,k}^p}$, i.e. future firm-specific and aggregate cumulative inflation rates have to be deflated by future cumulative target inflation rates.
  - However, under unit roots all **expected** future quarterly target inflation rates are simply equal to the current target rate.
  - In all **forward-looking** conditions like (19) and (20) we can therefore simply linearize around the current inflation target π_t^*.
  - Example: Our final conditions will contain **expected** inflation terms for periods t and $t+1$. For actual future inflation the correct definition is $\hat{\pi}_{t+1}^p = \ln(\pi_{t+1}^p) - \ln(\pi_{t+1}^*)$. But it is ok to linearize around π_t^* instead because only the **expected** future inflation deviation enters the equations, and for this we have (up to an approximation error that disappears in linearization):

$$E_t \hat{\pi}_{t+1}^p = E_t \left(\ln(\pi_{t+1}^p) - \ln(\pi_{t+1}^*) \right) = E_t \left(\ln(\pi_{t+1}^p) - \ln(\pi_t^*) \right)$$

  - The same is not true for **backward-looking** conditions. See Section 6 on "The Price Index".

5.4 Linearization

- Geometric distribution formulas:

$$\sum_{k=0}^{\infty} (\delta_p \beta)^k k = \frac{\delta_p \beta}{(1 - \delta_p \beta)^2} \tag{21}$$

$$\sum_{k=0}^{\infty} (\delta_p \beta)^k k^2 = \frac{\delta_p \beta (1 + \delta_p \beta)}{(1 - \delta_p \beta)^3} \tag{22}$$

- Linearization of goods demand:

$$\left(\hat{y}_{t+k}(j) - \hat{Y}_{t+k} \right) = -\bar{\sigma}_p \left(\hat{p}_t^p + k \hat{v}_t^p - \hat{\Pi}_{t,k}^p \right) \tag{23}$$

- Linearization of markup:

$$\hat{\mu}_t^p = \hat{\sigma}_t^p - \bar{\mu}_p \hat{\sigma}_t^p \tag{24}$$

5.4.1 Linearization for V_t^p

- Rewrite (19):

$$\begin{aligned} & p_t^p \left(\sigma_{t+k}^p - 1 \right) E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \check{\lambda}_{t+k} \check{y}_{t+k}(j) \left(\frac{(\check{v}_t^p)^k}{\check{\Pi}_{t,k}} \right) \\ = \; & E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \check{\lambda}_{t+k} \check{y}_{t+k}(j) \sigma_{t+k}^p \left(mc_{t+k} + \phi_p \left(\frac{\check{y}_{t+k}(j) - \check{Y}_{t+k}}{\check{Y}_{t+k}} \right) \right) \end{aligned}$$

- Linearization of the common terms $\check{\lambda}_{t+k} \check{y}_{t+k}(j)$ cancels.
- Remaining terms are:

$$\frac{\hat{p}_t^p}{1 - \delta_p \beta} + E_t \Sigma_{k=0}^{\infty} (\delta_p \beta)^k \left[k \hat{v}_t^p - \hat{\Pi}_{t,k}^p + \bar{\mu}_p \hat{\sigma}_{t+k}^p \right]$$

$$= E_t \Sigma_{k=0}^{\infty} (\delta_p \beta)^k \left[\widehat{mc}_{t+k} + \phi_p \bar{\mu}_p \left(\hat{y}_{t+k}(j) - \hat{Y}_{t+k} \right) + \hat{\sigma}_{t+k}^p \right]$$

- Using (21), (23) and (24):

$$\frac{\hat{p}_t^p}{1 - \delta_p \beta}(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p) + \frac{\hat{v}_t^p \delta_p \beta}{(1 - \delta_p \beta)^2}(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p) = E_t \Sigma_{k=0}^{\infty} (\delta_p \beta)^k \left[\widehat{mc}_{t+k} + \hat{\Pi}_{t,k}^p (1 + \phi_p \bar{\mu}_p \bar{\sigma}_p) + \hat{\mu}_{t+k}^p \right]$$

- Equivalently:

$$\frac{\hat{p}_t^p}{1 - \delta_p \beta} + \frac{\hat{v}_t^p \delta_p \beta}{(1 - \delta_p \beta)^2} = E_t \Sigma_{k=0}^{\infty} (\delta_p \beta)^k \left[\frac{\widehat{mc}_{t+k} + \hat{\mu}_{t+k}^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} + \hat{\Pi}_{t,k}^p \right] \tag{25}$$

5.4.2 Linearization for v_t^p

- Rewrite (20):

$$
\begin{aligned}
& p_t^p E_t \Sigma_{k=0}^{\infty} \left(\delta_p \beta\right)^k k \check{\lambda}_{t+k} \check{y}_{t+k}(j) (\sigma_{t+k}^p - 1) \left(\frac{(\check{v}_t^p)^k}{\check{\Pi}_{t,k}} \right) \\
= \quad & E_t \Sigma_{k=0}^{\infty} \left(\delta_p \beta\right)^k k \check{\lambda}_{t+k} \check{y}_{t+k}(j) \sigma_{t+k}^p \left(mc_{t+k} + \phi_p \left(\frac{\check{y}_{t+k}(j) - \check{Y}_{t+k}}{\check{Y}_{t+k}} \right) \right)
\end{aligned}
$$

- Linearization of the common terms $\check{\lambda}_{t+k} \check{y}_{t+k}(j)$ cancels.
- Applying (21) to the $\hat{p}_t^p$-term, the remaining terms are:

$$
\begin{aligned}
& \frac{\hat{p}_t^p \delta_p \beta}{\left(1 - \delta_p \beta\right)^2} + E_t \Sigma_{k=0}^{\infty} \left(\delta_p \beta\right)^k k \left[k \hat{v}_t^p - \hat{\Pi}_{t,k}^p + \bar{\mu}_p \hat{\sigma}_{t+k}^p \right] \\
= & E_t \Sigma_{k=0}^{\infty} \left(\delta_p \beta\right)^k k \left[\widehat{mc}_{t+k} + \phi_p \bar{\mu}_p \left(\hat{y}_{t+k}(j) - \hat{Y}_{t+k} \right) + \hat{\sigma}_{t+k}^p \right]
\end{aligned}
$$

- Applying (23) and (24), and simplifying, we get:

$$
\begin{aligned}
& \frac{\hat{p}_t^p \delta_p \beta}{\left(1 - \delta_p \beta\right)^2} (1 + \phi_p \bar{\mu}_p \bar{\sigma}_p) + E_t \Sigma_{k=0}^{\infty} \left(\delta_p \beta\right)^k k^2 \hat{v}_t^p (1 + \phi_p \bar{\mu}_p \bar{\sigma}_p) \\
= \quad & E_t \Sigma_{k=0}^{\infty} \left(\delta_p \beta\right)^k k \left[\widehat{mc}_{t+k} + \hat{\mu}_{t+k}^p + \hat{\Pi}_{t,k}^p (1 + \phi_p \bar{\mu}_p \bar{\sigma}_p) \right]
\end{aligned}
$$

- Applying (22) to the $\hat{v}_t^p$-term and simplifying further, we get:

$$
\frac{\hat{p}_t^p \delta_p \beta}{\left(1 - \delta_p \beta\right)^2} + \frac{\hat{v}_t^p \delta_p \beta (1 + \delta_p \beta)}{\left(1 - \delta_p \beta\right)^3} = E_t \Sigma_{k=0}^{\infty} \left(\delta_p \beta\right)^k k \left[\frac{\widehat{mc}_{t+k} + \hat{\mu}_{t+k}^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} + \hat{\Pi}_{t,k}^p \right] \tag{26}
$$

5.5 Quasi-Differencing

5.5.1 Quasi-Differencing for V_t^p-Equation (25)

- Rewrite (25) as:

$$\hat{p}_t^p + \frac{\hat{v}_t^p \delta_p \beta}{(1-\delta_p\beta)} = (1-\delta_p\beta) E_t \sum_{k=0}^{\infty} (\delta_p\beta)^k \left[\frac{\widehat{mc}_{t+k} + \hat{\mu}_{t+k}^p}{(1+\phi_p\bar{\mu}_p\bar{\sigma}_p)} + \hat{\Pi}_{t,k}^p \right] \tag{27}$$

- For future reference, lead this by one period and multiply by $(1-\delta_p\beta)$:

$$\begin{aligned} & (1-\delta_p\beta) E_t \hat{p}_{t+1}^p + \delta_p\beta E_t \hat{v}_{t+1}^p \\ = \quad & (1-\delta_p\beta)^2 E_t \left[\sum_{k=0}^{\infty} (\delta_p\beta)^k \frac{\widehat{mc}_{t+1+k} + \hat{\mu}_{t+1+k}^p}{(1+\phi_p\bar{\mu}_p\bar{\sigma}_p)} + \sum_{k=1}^{\infty} (\delta_p\beta)^k \hat{\Pi}_{t+1,k}^p \right] \end{aligned} \tag{28}$$

- Note the following for $\hat{\Pi}_{t+1,k}^p$:[1]

$$\begin{aligned} \hat{\Pi}_{t+1,k}^p &= 0 \text{ for } k=0, \\ &= \hat{\pi}_{t+2}^p \text{ for k=1} \\ &= \hat{\pi}_{t+2}^p + ... + \hat{\pi}_{t+k+1}^p \text{ for k=2,3,4,...} \end{aligned} \tag{29}$$

- Write out terms in (27):

$$\hat{p}_t^p + \frac{\hat{v}_t^p \delta_p\beta}{(1-\delta_p\beta)} = (1-\delta_p\beta) *$$

$$E_t \left[\frac{\widehat{mc}_t + \hat{\mu}_t^p}{(1+\phi_p\bar{\mu}_p\bar{\sigma}_p)} + (\delta_p\beta) \left(\frac{\widehat{mc}_{t+1} + \hat{\mu}_{t+1}^p}{(1+\phi_p\bar{\mu}_p\bar{\sigma}_p)} + \hat{\pi}_{t+1}^p \right) + (\delta_p\beta)^2 \left(\frac{\widehat{mc}_{t+2} + \hat{\mu}_{t+2}^p}{(1+\phi_p\bar{\mu}_p\bar{\sigma}_p)} + \hat{\pi}_{t+1}^p + \hat{\pi}_{t+2}^p \right) \right.$$
$$\left. + (\delta_p\beta)^3 \left(\frac{\widehat{mc}_{t+3} + \hat{\mu}_{t+3}^p}{(1+\phi_p\bar{\mu}_p\bar{\sigma}_p)} + \hat{\pi}_{t+1}^p + \hat{\pi}_{t+2}^p + \hat{\pi}_{t+3}^p \right) + ... \right]$$

- Multiply the last equation by $\delta_p\beta$, and lead by one period:

$$\delta_p\beta E_t \hat{p}_{t+1}^p + \frac{(\delta_p\beta)^2}{(1-\delta_p\beta)} E_t \hat{v}_{t+1}^p = (1-\delta_p\beta) *$$

$$E_t \left[(\delta_p\beta) \frac{\widehat{mc}_{t+1} + \hat{\mu}_{t+1}^p}{(1+\phi_p\bar{\mu}_p\bar{\sigma}_p)} + (\delta_p\beta)^2 \left(\frac{\widehat{mc}_{t+2} + \hat{\mu}_{t+2}^p}{(1+\phi_p\bar{\mu}_p\bar{\sigma}_p)} + \hat{\pi}_{t+2}^p \right) \right.$$
$$\left. + (\delta_p\beta)^3 \left(\frac{\widehat{mc}_{t+3} + \hat{\mu}_{t+3}^p}{(1+\phi_p\bar{\mu}_p\bar{\sigma}_p)} + \hat{\pi}_{t+2}^p + \hat{\pi}_{t+3}^p \right) + ... \right]$$

[1] The first line shows why for the last term in the previous equation we can let the subscript run from 1 instead of 0.

- Deduct the last equation from the preceding one:

$$\left[\hat{p}_t^p - \delta_p \beta E_t \hat{p}_{t+1}^p\right] + \frac{\delta_p \beta}{(1-\delta_p \beta)} \left[\hat{v}_t^p - \delta_p \beta E_t \hat{v}_{t+1}^p\right]$$
$$= (1-\delta_p \beta) \left[\frac{\widehat{mc}_t + \hat{\mu}_t^p}{(1+\phi_p \bar{\mu}_p \bar{\sigma}_p)} + E_t \hat{\pi}_{t+1}^p \left(\delta_p \beta + (\delta_p \beta)^2 + + (\delta_p \beta)^3 + ...\right)\right]$$

- Equivalently:

$$\left[\hat{p}_t^p - E_t \hat{p}_{t+1}^p + (1-\delta_p \beta) E_t \hat{p}_{t+1}^p\right] + \frac{\delta_p \beta}{(1-\delta_p \beta)} \left[\hat{v}_t^p - E_t \hat{v}_{t+1}^p + (1-\delta_p \beta) E_t \hat{v}_{t+1}^p\right]$$
$$= (1-\delta_p \beta) \frac{\widehat{mc}_t + \hat{\mu}_t^p}{(1+\phi_p \bar{\mu}_p \bar{\sigma}_p)} + \delta_p \beta E_t \hat{\pi}_{t+1}^p$$

- Equivalently:

$$\left[\hat{p}_t^p - E_t \hat{p}_{t+1}^p\right] + \frac{\delta_p \beta}{(1-\delta_p \beta)} \left[\hat{v}_t^p - E_t \hat{v}_{t+1}^p\right]$$
$$= \quad (1-\delta_p \beta) \left(\frac{\widehat{mc}_t + \hat{\mu}_t^p}{(1+\phi_p \bar{\mu}_p \bar{\sigma}_p)} - E_t \hat{p}_{t+1}^p\right) + \delta_p \beta \left(E_t \hat{\pi}_{t+1}^p - E_t \hat{v}_{t+1}^p\right)$$

- Equivalently:

$$\left[E_t \hat{p}_{t+1}^p - \hat{p}_t^p\right] + \frac{\delta_p \beta}{(1-\delta_p \beta)} \left[E_t \hat{v}_{t+1}^p - \hat{v}_t^p\right]$$
$$= \quad -(1-\delta_p \beta) \left(\frac{\widehat{mc}_t + \hat{\mu}_t^p}{(1+\phi_p \bar{\mu}_p \bar{\sigma}_p)} - E_t \hat{p}_{t+1}^p\right) + \delta_p \beta \left(E_t \hat{v}_{t+1}^p - E_t \hat{\pi}_{t+1}^p\right)$$

- Equivalently:

$$\delta_p \beta E_t \hat{p}_{t+1}^p - \hat{p}_t^p + \frac{\delta_p \beta}{(1-\delta_p \beta)} \left[E_t \hat{v}_{t+1}^p - \hat{v}_t^p\right] \tag{30}$$
$$= \quad -(1-\delta_p \beta) \left(\frac{\widehat{mc}_t + \hat{\mu}_t^p}{(1+\phi_p \bar{\mu}_p \bar{\sigma}_p)}\right) + \delta_p \beta E_t \hat{v}_{t+1}^p - \delta_p \beta E_t \hat{\pi}_{t+1}^p$$

5.5.2 Quasi-Differencing for v_t^p-Equation (26)

- Rewrite (26) as:

$$\hat{p}_t^p + \frac{(1+\delta_p\beta)}{(1-\delta_p\beta)}\hat{v}_t^p = \frac{(1-\delta_p\beta)^2}{\delta_p\beta} E_t \sum_{k=0}^{\infty} (\delta_p\beta)^k\, k \left[\frac{\widehat{mc}_{t+k} + \hat{\mu}_{t+k}^p}{(1+\phi_p\bar{\mu}_p\bar{\sigma}_p)} + \hat{\Pi}_{t,k}^p\right]$$

- Write out terms:

$$\hat{p}_t^p + \frac{(1+\delta_p\beta)}{(1-\delta_p\beta)}\hat{v}_t^p = \left(\frac{(1-\delta_p\beta)^2}{\delta_p\beta}\right) *$$

$$E_t\left[(\delta_p\beta)\left(\frac{\widehat{mc}_{t+1} + \hat{\mu}_{t+1}^p}{(1+\phi_p\bar{\mu}_p\bar{\sigma}_p)} + \hat{\pi}_{t+1}^p\right) + 2(\delta_p\beta)^2\left(\frac{\widehat{mc}_{t+2} + \hat{\mu}_{t+2}^p}{(1+\phi_p\bar{\mu}_p\bar{\sigma}_p)} + \hat{\pi}_{t+1}^p + \hat{\pi}_{t+2}^p\right)\right.$$
$$\left. +3(\delta_p\beta)^3\left(\frac{\widehat{mc}_{t+3} + \hat{\mu}_{t+3}^p}{(1+\phi_p\bar{\mu}_p\bar{\sigma}_p)} + \hat{\pi}_{t+1}^p + \hat{\pi}_{t+2}^p + \hat{\pi}_{t+3}^p\right) + ...\right]$$

- Multiply the last equation by $\delta_p\beta$, and lead by one period:

$$\delta_p\beta E_t\hat{p}_{t+1}^p + \frac{\delta_p\beta(1+\delta_p\beta)}{(1-\delta_p\beta)}E_t\hat{v}_{t+1}^p = \left(\frac{(1-\delta_p\beta)^2}{\delta_p\beta}\right) *$$

$$E_t\left[(\delta_p\beta)^2\left(\frac{\widehat{mc}_{t+2} + \hat{\mu}_{t+2}^p}{(1+\phi_p\bar{\mu}_p\bar{\sigma}_p)} + \hat{\pi}_{t+2}^p\right) + 2(\delta_p\beta)^3\left(\frac{\widehat{mc}_{t+3} + \hat{\mu}_{t+3}^p}{(1+\phi_p\bar{\mu}_p\bar{\sigma}_p)} + \hat{\pi}_{t+2}^p + \hat{\pi}_{t+3}^p\right) + ...\right]$$

- Deduct the last equation from the preceding one:

$$\left[\hat{p}_t^p - \delta_p\beta E_t\hat{p}_{t+1}^p\right] + \frac{(1+\delta_p\beta)}{(1-\delta_p\beta)}\left[\hat{v}_t^p - \delta_p\beta E_t\hat{v}_{t+1}^p\right] = \left(\frac{(1-\delta_p\beta)^2}{\delta_p\beta}\right) *$$

$$E_t\left[(\delta_p\beta)\frac{\widehat{mc}_{t+1} + \hat{\mu}_{t+1}^p}{(1+\phi_p\bar{\mu}_p\bar{\sigma}_p)} + (\delta_p\beta)^2\left(\frac{\widehat{mc}_{t+2} + \hat{\mu}_{t+2}^p}{(1+\phi_p\bar{\mu}_p\bar{\sigma}_p)} + \hat{\pi}_{t+2}^p\right) + (\delta_p\beta)^3\left(\frac{\widehat{mc}_{t+3} + \hat{\mu}_{t+3}^p}{(1+\phi_p\bar{\mu}_p\bar{\sigma}_p)} + \hat{\pi}_{t+2}^p + \hat{\pi}_{t+3}^p\right) + ...\right.$$
$$\left. +\hat{\pi}_{t+1}^p\left(\delta_p\beta + 2(\delta_p\beta)^2 + 3(\delta_p\beta)^3 + ...\right)\right]$$

- Use (21) for the final term:

$$E_t\hat{\pi}_{t+1}^p\left(\frac{(1-\delta_p\beta)^2}{\delta_p\beta}\right)\left(\delta_p\beta + 2(\delta_p\beta)^2 + 3(\delta_p\beta)^3 + ...\right) = E_t\hat{\pi}_{t+1}^p$$

- Simplify:

$$\left[\hat{p}_t^p - \delta_p\beta E_t\hat{p}_{t+1}^p\right] + \frac{(1+\delta_p\beta)}{(1-\delta_p\beta)}\left[\hat{v}_t^p - \delta_p\beta E_t\hat{v}_{t+1}^p\right] = E_t\hat{\pi}_{t+1}^p + \left((1-\delta_p\beta)^2\right) *$$

$$E_t\left[\frac{\widehat{mc}_{t+1} + \hat{\mu}_{t+1}^p}{(1+\phi_p\bar{\mu}_p\bar{\sigma}_p)} + (\delta_p\beta)\left(\frac{\widehat{mc}_{t+2} + \hat{\mu}_{t+2}^p}{(1+\phi_p\bar{\mu}_p\bar{\sigma}_p)} + \hat{\pi}_{t+2}^p\right) + (\delta_p\beta)^2\left(\frac{\widehat{mc}_{t+3} + \hat{\mu}_{t+3}^p}{(1+\phi_p\bar{\mu}_p\bar{\sigma}_p)} + \hat{\pi}_{t+2}^p + \hat{\pi}_{t+3}^p\right) + ...\right]$$

- Rewrite:

$$\left[\hat{p}_t^p - \delta_p \beta E_t \hat{p}_{t+1}^p\right] + \frac{(1+\delta_p\beta)}{(1-\delta_p\beta)}\left[\hat{v}_t^p - \delta_p\beta E_t \hat{v}_{t+1}^p\right] = E_t \hat{\pi}_{t+1}^p + \left((1-\delta_p\beta)^2\right) *$$

$$\left[E_t \sum_{k=0}^{\infty} (\delta_p\beta)^k \frac{\widehat{mc}_{t+1+k} + \hat{\mu}_{t+1+k}^p}{(1+\phi_p\bar{\mu}_p\bar{\sigma}_p)} + E_t\left[(\delta_p\beta)\,\hat{\pi}_{t+2}^p + (\delta_p\beta)^2\left(\hat{\pi}_{t+2}^p + \hat{\pi}_{t+3}^p\right) + ...\right]\right]$$

- The final term can be rewritten using (29):

$$E_t\left[(\delta_p\beta)\,\hat{\pi}_{t+2}^p + (\delta_p\beta)^2\left(\hat{\pi}_{t+2}^p + \hat{\pi}_{t+3}^p\right) + ...\right] = \sum_{k=1}^{\infty} (\delta_p\beta)^k\,\hat{\Pi}_{t+1,k}^p$$

- Then we have:

$$\left[\hat{p}_t^p - \delta_p \beta E_t \hat{p}_{t+1}^p\right] + \frac{(1+\delta_p\beta)}{(1-\delta_p\beta)}\left[\hat{v}_t^p - \delta_p\beta E_t \hat{v}_{t+1}^p\right] =$$

$$E_t \hat{\pi}_{t+1}^p + \left((1-\delta_p\beta)^2\right) E_t \left[\sum_{k=0}^{\infty} (\delta_p\beta)^k \frac{\widehat{mc}_{t+1+k} + \hat{\mu}_{t+1+k}^p}{(1+\phi_p\bar{\mu}_p\bar{\sigma}_p)} + \sum_{k=1}^{\infty} (\delta_p\beta)^k\,\hat{\Pi}_{t+1,k}^p\right]$$

- Now we can replace the right-hand side using (28):

$$\begin{aligned} &\left[\hat{p}_t^p - \delta_p \beta E_t \hat{p}_{t+1}^p\right] + \frac{(1+\delta_p\beta)}{(1-\delta_p\beta)}\left[\hat{v}_t^p - \delta_p\beta E_t \hat{v}_{t+1}^p\right] \\ = \quad & E_t \hat{\pi}_{t+1}^p + (1-\delta_p\beta)\, E_t \hat{p}_{t+1}^p + \delta_p\beta E_t \hat{v}_{t+1}^p \end{aligned}$$

- Further simplification:

$$\left[\hat{p}_t^p - E_t \hat{p}_{t+1}^p + (1-\delta_p\beta)\, E_t \hat{p}_{t+1}^p\right] + \frac{(1+\delta_p\beta)}{(1-\delta_p\beta)}\left[\hat{v}_t^p - E_t \hat{v}_{t+1}^p + (1-\delta_p\beta) E_t \hat{v}_{t+1}^p\right]$$

$$= E_t \hat{\pi}_{t+1}^p + (1-\delta_p\beta)\, E_t \hat{p}_{t+1}^p + \delta_p\beta E_t \hat{v}_{t+1}^p$$

- Cancel terms:

$$\begin{aligned} &\left[\hat{p}_t^p - E_t \hat{p}_{t+1}^p\right] + \frac{(1+\delta_p\beta)}{(1-\delta_p\beta)}\left[\hat{v}_t^p - E_t \hat{v}_{t+1}^p\right] \\ = \quad & E_t \hat{\pi}_{t+1}^p - E_t \hat{v}_{t+1}^p \end{aligned}$$

- Equivalently:

$$\begin{aligned} &\left[E_t \hat{p}_{t+1}^p - \hat{p}_t^p\right] + \frac{(1+\delta_p\beta)}{(1-\delta_p\beta)}\left[E_t \hat{v}_{t+1}^p - \hat{v}_t^p\right] \\ = \quad & E_t \hat{v}_{t+1}^p - E_t \hat{\pi}_{t+1}^p \end{aligned}$$

- Equivalently:

$$E_t\hat{p}^p_{t+1} = \hat{p}^p_t + E_t\hat{v}^p_{t+1} - E_t\hat{\pi}^p_{t+1} - \frac{(1+\delta_p\beta)}{(1-\delta_p\beta)}\left[E_t\hat{v}^p_{t+1} - \hat{v}^p_t\right] \tag{31}$$

- Equivalently:

$$E_t\hat{p}^p_{t+1} - \hat{p}^p_t = \frac{-2\delta_p\beta}{(1-\delta_p\beta)}E_t\hat{v}^p_{t+1} - E_t\hat{\pi}^p_{t+1} + \frac{(1+\delta_p\beta)}{(1-\delta_p\beta)}\hat{v}^p_t \tag{32}$$

5.5.3 Combine (30) and (31)

- (30) for ease of reference:

$$\begin{aligned} &\delta_p\beta E_t\hat{p}^p_{t+1} - \hat{p}^p_t + \frac{\delta_p\beta}{(1-\delta_p\beta)}\left[E_t\hat{v}^p_{t+1} - \hat{v}^p_t\right] \\ = &-(1-\delta_p\beta)\left(\frac{\widehat{mc}_t + \hat{\mu}^p_t}{(1+\phi_p\bar{\mu}_p\bar{\sigma}_p)}\right) + \delta_p\beta E_t\hat{v}^p_{t+1} - \delta_p\beta E_t\hat{\pi}^p_{t+1} \end{aligned}$$

- Plug in (31):

$$\begin{aligned} &\delta_p\beta\left[\hat{p}^p_t + E_t\hat{v}^p_{t+1} - E_t\hat{\pi}^p_{t+1} - \frac{(1+\delta_p\beta)}{(1-\delta_p\beta)}\left[E_t\hat{v}^p_{t+1} - \hat{v}^p_t\right]\right] \\ &-\hat{p}^p_t + \frac{\delta_p\beta}{(1-\delta_p\beta)}\left[E_t\hat{v}^p_{t+1} - \hat{v}^p_t\right] \\ &= -(1-\delta_p\beta)\left(\frac{\widehat{mc}_t + \hat{\mu}^p_t}{(1+\phi_p\bar{\mu}_p\bar{\sigma}_p)}\right) + \delta_p\beta E_t\hat{v}^p_{t+1} - \delta_p\beta E_t\hat{\pi}^p_{t+1} \end{aligned}$$

- Equivalently:

$$\begin{aligned} &(\delta_p\beta - 1)\hat{p}^p_t + \delta_p\beta E_t\hat{v}^p_{t+1} - \delta_p\beta E_t\hat{\pi}^p_{t+1} + \frac{\delta_p\beta}{(1-\delta_p\beta)}(1-1-\delta_p\beta)\left(E_t\hat{v}^p_{t+1} - \hat{v}^p_t\right) \\ &= -(1-\delta_p\beta)\left(\frac{\widehat{mc}_t + \hat{\mu}^p_t}{(1+\phi_p\bar{\mu}_p\bar{\sigma}_p)}\right) + \delta_p\beta E_t\hat{v}^p_{t+1} - \delta_p\beta E_t\hat{\pi}^p_{t+1} \end{aligned}$$

- Cancel terms and multiply by -1:

$$\begin{aligned} &(1-\delta_p\beta)\hat{p}^p_t + \frac{(\delta_p\beta)^2}{(1-\delta_p\beta)}\left(E_t\hat{v}^p_{t+1} - \hat{v}^p_t\right) \\ &= (1-\delta_p\beta)\left(\frac{\widehat{mc}_t + \hat{\mu}^p_t}{(1+\phi_p\bar{\mu}_p\bar{\sigma}_p)}\right) \end{aligned}$$

- Simplify further:

$$\frac{(\delta_p\beta)^2}{(1-\delta_p\beta)}\left(E_t\hat{v}^p_{t+1} - \hat{v}^p_t\right) = (1-\delta_p\beta)\left(\frac{\widehat{mc}_t + \hat{\mu}^p_t}{(1+\phi_p\bar{\mu}_p\bar{\sigma}_p)} - \hat{p}^p_t\right)$$

- Preliminary difference equation for $\hat{v}^p_t$:

$$\left(E_t\hat{v}^p_{t+1} - \hat{v}^p_t\right) = \frac{(1-\delta_p\beta)^2}{(\delta_p\beta)^2}\left(\frac{\widehat{mc}_t + \hat{\mu}^p_t}{(1+\phi_p\bar{\mu}_p\bar{\sigma}_p)} - \hat{p}^p_t\right) \tag{33}$$

6 THE PRICE INDEX

6.1 Formula for the Index

- Price Level:

$$P_t = \left[(1-\delta_p) \sum_{s=0}^{\infty} \delta_p^s \left[V_{t-s}^p (v_{t-s}^p)^s \right]^{1-\sigma_t^p} \right]^{\frac{1}{1-\sigma_t^p}}$$

- Deflate by current target price level P_t^* and write out terms ($\check{P}_t = P_t/P_t^*$, $\check{V}_t^p = V_t^p/P_t^*$):

$$\left(\check{P}_t\right)^{1-\sigma_t^p} = (1-\delta_p)\left(\check{V}_t^p\right)^{1-\sigma_t^p} + (1-\delta_p)\delta_p \left(\check{V}_{t-1}^p\right)^{1-\sigma_t^p} \left(\frac{v_{t-1}^p}{\pi_t^*}\right)^{1-\sigma_t^p}$$

$$+(1-\delta_p)\delta_p^2 \left(\check{V}_{t-2}^p\right)^{1-\sigma_t^p} \left(\frac{\left(\hat{v}_{t-2}^{pp}\right)^2}{\pi_t^* \pi_{t-1}^*}\right)^{(1-\sigma_t^p)} + (1-\delta_p)\delta_p^3 \left(\check{V}_{t-3}^p\right)^{1-\sigma_t^p} \left(\frac{\left(v_{t-3}^p\right)^3}{\pi_t^* \pi_{t-1}^* \pi_{t-2}^*}\right)^{(1-\sigma_t^p)} + \ldots$$

- Divide by $\check{P}_{t-1}$:

$$\begin{aligned}
\left(\frac{\check{P}_t}{\check{P}_{t-1}}\right)^{1-\sigma_t^p} &= (1-\delta_p)\left(\frac{\check{V}_t^p}{\check{P}_t}\right)^{1-\sigma_t^p} \left(\frac{\check{P}_t}{\check{P}_{t-1}}\right)^{1-\sigma_t^p} \\
&\quad + (1-\delta_p)\delta_p \left(\frac{\check{V}_{t-1}^p}{\check{P}_{t-1}}\right)^{1-\sigma_t^p} \left(\frac{v_{t-1}^p}{\pi_t^*}\right)^{1-\sigma_t^p} \\
&\quad + (1-\delta_p)\delta_p^2 \left(\frac{\check{V}_{t-2}^p}{\check{P}_{t-2}}\right)^{1-\sigma_t^p} \left(\frac{\check{P}_{t-2}}{\check{P}_{t-1}}\right)^{1-\sigma_t^p} \left(\frac{\left(v_{t-2}^p\right)^2}{\pi_t^* \pi_{t-1}^*}\right)^{(1-\sigma_t^p)} \\
&\quad + (1-\delta_p)\delta_p^3 \left(\frac{\check{V}_{t-3}^p}{\check{P}_{t-3}}\right)^{1-\sigma_t^p} \left(\frac{\check{P}_{t-3}}{\check{P}_{t-2}}\right)^{1-\sigma_t^p} \left(\frac{\check{P}_{t-2}}{\check{P}_{t-1}}\right)^{1-\sigma_t^p} \left(\frac{\left(v_{t-3}^p\right)^3}{\pi_t^* \pi_{t-1}^* \pi_{t-2}^*}\right)^{(1-\sigma_t^p)} + \ldots
\end{aligned}$$

- Use $\check{P}_t/\check{P}_{t-1} = \check{\pi}_t^p = \frac{\pi_t^p}{\pi_t^*}$ (this is the deviation of gross inflation from its target):

$$\begin{aligned}
\left(\check{\pi}_t^p\right)^{1-\sigma_t^p} &= (1-\delta_p)\left(p_t\right)^{1-\sigma_t^p}\left(\check{\pi}_t^p\right)^{1-\sigma_t^p} \\
&+(1-\delta_p)\delta_p\left(p_{t-1}\right)^{1-\sigma_t^p}\left(\frac{v_{t-1}^p}{\pi_t^*}\right)^{1-\sigma_t^p} \\
&+(1-\delta_p)\delta_p^2\left(p_{t-2}\right)^{1-\sigma_t^p}\left(\frac{1}{\check{\pi}_{t-1}^p}\right)^{1-\sigma_t^p}\left(\frac{\left(v_{t-2}^p\right)^p}{\pi_t^*\pi_{t-1}^*}\right)^{(1-\sigma_t^p)} \\
&+(1-\delta_p)\delta_p^3\left(p_{t-3}\right)^{1-\sigma_t^p}\left(\frac{1}{\check{\pi}_{t-2}^p}\right)^{1-\sigma_t^p}\left(\frac{1}{\check{\pi}_{t-1}^p}\right)^{1-\sigma_t^p}\left(\frac{\left(v_{t-3}^p\right)^3}{\pi_t^*\pi_{t-1}^*\pi_{t-2}^*}\right)^{(1-\sigma_t^p)}+...
\end{aligned}$$

- Divide through by $\left(\check{\pi}_t^p\right)^{1-\sigma_t^p}$:

$$\begin{aligned}
1 &= (1-\delta_p)p_t^{1-\sigma_t^p} \\
&+(1-\delta_p)\delta_p p_{t-1}^{1-\sigma_t^p}\left(\frac{v_{t-1}^p}{\check{\pi}_t^p\pi_t^*}\right)^{1-\sigma_t^p} \\
&+(1-\delta_p)\delta_p^2 p_{t-2}^{1-\sigma_t^p}\left(\frac{\left(v_{t-2}^p\right)^2}{\check{\pi}_{t-1}^p\check{\pi}_t^p\pi_{t-1}^*\pi_t^*}\right)^{1-\sigma_t^p} \\
&+(1-\delta_p)\delta_p^3 p_{t-3}^{1-\sigma_t^p}\left(\frac{\left(v_{t-3}^p\right)^3}{\check{\pi}_{t-2}^p\check{\pi}_{t-1}^p\check{\pi}_t^p\pi_{t-2}^*\pi_{t-1}^*\pi_t^*}\right)^{1-\sigma_t^p}+...
\end{aligned}$$

- Let $\check{v}_{t-1}^p = v_{t-1}^p/\pi_{t-1}^*$, the deviation of firm-specific gross inflation from the inflation target, and use $\pi_t^*/\pi_{t-1}^* = \varepsilon_t^{\pi^*}$:

$$\begin{aligned}
1 &= (1-\delta_p)p_t^{1-\sigma_t^p} \qquad (34)\\
&+(1-\delta_p)\delta_p p_{t-1}^{1-\sigma_t^p}\left(\frac{\check{v}_{t-1}^p}{\check{\pi}_t^p\varepsilon_t^{\pi^*}}\right)^{1-\sigma_t^p} \\
&+(1-\delta_p)\delta_p^2 p_{t-2}^{1-\sigma_t^p}\left(\frac{\left(\check{v}_{t-2}^p\right)^2}{\check{\pi}_{t-1}^p\check{\pi}_t^p\left(\varepsilon_{t-1}^{\pi^*}\right)^2\varepsilon_t^{\pi^*}}\right)^{1-\sigma_t^p} \\
&+(1-\delta_p)\delta_p^3 p_{t-3}^{1-\sigma_t^p}\left(\frac{\left(\check{v}_{t-3}^p\right)^3}{\check{\pi}_{t-2}^p\check{\pi}_{t-1}^p\check{\pi}_t^p\left(\varepsilon_{t-2}^{\pi^*}\right)^3\left(\varepsilon_{t-1}^{\pi^*}\right)^2\varepsilon_t^{\pi^*}}\right)^{1-\sigma_t^p}+...
\end{aligned}$$

6.2 Linearize (34)

- Linearize:

$$
\begin{aligned}
0 &= (1-\delta_p)(1-\sigma_t^p)\hat{p}_t^p \\
&+(1-\delta_p)(1-\sigma_t^p)\delta_p\left(\hat{p}_{t-1}^p+\hat{v}_{t-1}^p-\hat{\pi}_t^p-\hat{\varepsilon}_t^{\pi^*}\right) \\
&+(1-\delta_p)(1-\sigma_t^p)\delta_p^2\left(\hat{p}_{t-2}^p+2\hat{v}_{t-2}^p-\hat{\pi}_{t-1}^p-\hat{\pi}_t^p-2\hat{\varepsilon}_{t-1}^{\pi^*}-\hat{\varepsilon}_t^{\pi^*}\right) \\
&+(1-\delta_p)(1-\sigma_t^p)\delta_p^3\left(\hat{p}_{t-3}^p+3\hat{v}_{t-3}^p-\hat{\pi}_{t-2}^p-\hat{\pi}_{t-1}^p-\hat{\pi}_t^p-3\hat{\varepsilon}_{t-2}^{\pi^*}-2\hat{\varepsilon}_{t-1}^{\pi^*}-\hat{\varepsilon}_t^{\pi^*}\right)+...
\end{aligned}
$$

- Cancel terms and bring $\hat{\pi}_t^p$ onto left-hand side:

$$
\hat{\pi}_t^p\left(\delta_p+\delta_p^2+\delta_p^3+...\right)=\hat{p}_t^p+\delta_p\left(\hat{p}_{t-1}^p+\hat{v}_{t-1}^p-\hat{\varepsilon}_t^{\pi^*}\right)
$$

$$
+\delta_p^2\left(\hat{p}_{t-2}^p+2\hat{v}_{t-2}^p-\hat{\pi}_{t-1}^p-2\hat{\varepsilon}_{t-1}^{\pi^*}-\hat{\varepsilon}_t^{\pi^*}\right)+\delta_p^3\left(\hat{p}_{t-3}^p+3\hat{v}_{t-3}^p-\hat{\pi}_{t-2}^p-\hat{\pi}_{t-1}^p-3\hat{\varepsilon}_{t-2}^{\pi^*}-2\hat{\varepsilon}_{t-1}^{\pi^*}-\hat{\varepsilon}_t^{\pi^*}\right)+...
$$

- Equivalently:

$$
\hat{\pi}_t^p=\frac{1-\delta_p}{\delta_p}\hat{p}_t^p+(1-\delta_p)\left(\hat{p}_{t-1}^p+\hat{v}_{t-1}^p-\hat{\varepsilon}_t^{\pi^*}\right)+(1-\delta_p)\delta_p\left(\hat{p}_{t-2}^p+2\hat{v}_{t-2}^p-\hat{\pi}_{t-1}^p-2\hat{\varepsilon}_{t-1}^{\pi^*}-\hat{\varepsilon}_t^{\pi^*}\right)
$$

$$
+(1-\delta_p)\delta_p^2\left(\hat{p}_{t-3}^p+3\hat{v}_{t-3}^p-\hat{\pi}_{t-2}^p-\hat{\pi}_{t-1}^p-3\hat{\varepsilon}_{t-2}^{\pi^*}-2\hat{\varepsilon}_{t-1}^{\pi^*}-\hat{\varepsilon}_t^{\pi^*}\right)+...
$$

6.3 Quasi-Differencing

- Lag the last equation and multiply by δ_p:

$$
\begin{aligned}
\delta_p\hat{\pi}_{t-1}^p &= (1-\delta_p)\hat{p}_{t-1}^p+(1-\delta_p)\delta_p\left(\hat{p}_{t-2}^p+\hat{v}_{t-2}^p-\hat{\varepsilon}_{t-1}^{\pi^*}\right) \\
&+(1-\delta_p)\delta_p^2\left(\hat{p}_{t-3}^p+2\hat{v}_{t-3}^p-\hat{\pi}_{t-2}^p-2\hat{\varepsilon}_{t-2}^{\pi^*}-\hat{\varepsilon}_{t-1}^{\pi^*}\right)+...
\end{aligned}
$$

- Deduct the last equation from the preceding one:

$$
\begin{aligned}
\hat{\pi}_t^p-\delta_p\hat{\pi}_{t-1}^p &= \frac{1-\delta_p}{\delta_p}\hat{p}_t^p+(1-\delta_p)\left(\hat{v}_{t-1}^p-\hat{\varepsilon}_t^{\pi^*}\right)+(1-\delta_p)\delta_p\left(\hat{v}_{t-2}^p-\hat{\pi}_{t-1}^p-\hat{\varepsilon}_{t-1}^{\pi^*}-\hat{\varepsilon}_t^{\pi^*}\right) \\
&+(1-\delta_p)\delta_p^2\left(\hat{v}_{t-3}^p-\hat{\pi}_{t-1}^p-\hat{\varepsilon}_{t-2}^{\pi^*}-\hat{\varepsilon}_{t-1}^{\pi^*}-\hat{\varepsilon}_t^{\pi^*}\right)+...
\end{aligned}
$$

- Equivalently:

$$
\begin{aligned}
\hat{\pi}_t^p-\delta_p\hat{\pi}_{t-1}^p &= \frac{1-\delta_p}{\delta_p}\hat{p}_t^p-\delta_p\hat{\pi}_{t-1}^p+(1-\delta_p)\left(\hat{v}_{t-1}^p-\hat{\varepsilon}_t^{\pi^*}\right)+(1-\delta_p)\delta_p\left(\hat{v}_{t-2}^p-\hat{\varepsilon}_{t-1}^{\pi^*}-\hat{\varepsilon}_t^{\pi^*}\right) \\
&+(1-\delta_p)\delta_p^2\left(\hat{v}_{t-3}^p-\hat{\varepsilon}_{t-2}^{\pi^*}-\hat{\varepsilon}_{t-1}^{\pi^*}-\hat{\varepsilon}_t^{\pi^*}\right)+...
\end{aligned}
$$

- Equivalently:

$$
\begin{aligned}
\hat{\pi}_t^p &= \frac{1-\delta_p}{\delta_p}\hat{p}_t^p+(1-\delta_p)\left(\hat{v}_{t-1}^p-\hat{\varepsilon}_t^{\pi^*}\right)+(1-\delta_p)\delta_p\left(\hat{v}_{t-2}^p-\hat{\varepsilon}_{t-1}^{\pi^*}-\hat{\varepsilon}_t^{\pi^*}\right) \\
&+(1-\delta_p)\delta_p^2\left(\hat{v}_{t-3}^p-\hat{\varepsilon}_{t-2}^{\pi^*}-\hat{\varepsilon}_{t-1}^{\pi^*}-\hat{\varepsilon}_t^{\pi^*}\right)+...
\end{aligned}
$$

- We finally obtain the key expression discussed at some length in the paper:

$$\hat{\pi}_t^p = \frac{1-\delta_p}{\delta_p}\hat{p}_t^p + \hat{\psi}_t^p \tag{35}$$

$$\hat{\psi}_t^p = (1-\delta_p)\left(\hat{v}_{t-1}^p - \hat{\varepsilon}_t^{\pi^*}\right) + (1-\delta_p)\,\delta_p\left(\hat{v}_{t-2}^p - \hat{\varepsilon}_{t-1}^{\pi^*} - \hat{\varepsilon}_t^{\pi^*}\right) + (1-\delta_p)\,\delta_p^2\left(\hat{v}_{t-3}^p - \hat{\varepsilon}_{t-2}^{\pi^*} - \hat{\varepsilon}_{t-1}^{\pi^*} - \hat{\varepsilon}_t^{\pi^*}\right) + ...$$

6.4 The Auxiliary Variable $\hat{\psi}_t^p$

- Lag the last equation and multiply by δ_p:

$$\delta_p\hat{\psi}_{t-1}^p = (1-\delta_p)\delta_p\left(\hat{v}_{t-2}^p - \hat{\varepsilon}_{t-1}^{\pi^*}\right) + (1-\delta_p)\delta_p^2\left(\hat{v}_{t-3}^p - \hat{\varepsilon}_{t-2}^{\pi^*} - \hat{\varepsilon}_{t-1}^{\pi^*}\right) + ...$$

- Deduct the last equation from the preceding one:

$$\hat{\psi}_t^p = \delta_p\hat{\psi}_{t-1}^p + (1-\delta_p)\hat{v}_{t-1}^p - \hat{\varepsilon}_t^{\pi^*} \tag{36}$$

- Also, for future reference:

$$\left(E_t\hat{\psi}_{t+1}^p - \hat{\psi}_t^p\right) = (1-\delta_p)\hat{v}_t^p - (1-\delta_p)\,\hat{\psi}_t^p \tag{37}$$

- Finally we will also be able to use (35) to substitute $\hat{p}_t^p$ out of equation (33):

$$\hat{p}_t^p = \frac{\delta_p}{1-\delta_p}\left(\hat{\pi}_t^p - \hat{\psi}_t^p\right) \tag{38}$$

6.5 More Intuition for (36)

- Assume that we incorrectly linearize (34) around the period t inflation target π_t^* regardless of the time subscript of the variables. We get the following, after changing notation to distinguish this linearization from (36):

$$\widehat{\psi}_t = \delta_p\widehat{\psi_{t-1}} + (1-\delta_p)\widehat{v_{t-1}^p} \tag{39}$$

- Note that for $\widehat{v_{t-1}^p}$ we have (the same holds for $\widehat{\psi_{t-1}}$)

$$\begin{aligned}\widehat{v_{t-1}^p} &= \ln(v_{t-1}^p) - \ln(\pi_t^*) \quad \text{, while} \\ \hat{v}_{t-1}^p &= \ln(v_{t-1}^p) - \ln(\pi_{t-1}^*)\end{aligned}$$

- This implies that (again similarly for $\widehat{\psi_{t-1}}$)

$$\widehat{v_{t-1}^p} = \hat{v}_{t-1}^p - \hat{\varepsilon}_t^{\pi^*}$$

- Also

$$\widehat{\psi}_t = \hat{\psi}_t^p$$

- Substituting the foregoing into (39) we obtain (36):

$$\hat{\psi}_t^p = \delta_p\hat{\psi}_{t-1}^p + (1-\delta_p)\hat{v}_{t-1}^p - \hat{\varepsilon}_t^{\pi^*}$$

7 FINAL INFLATION DYNAMICS

7.1 Final Equation for $\hat{v}_t^p$

- Equation (33) reproduced for ease of reference:

$$\left(E_t \hat{v}_{t+1}^p - \hat{v}_t^p\right) = \frac{(1-\delta_p\beta)^2}{(\delta_p\beta)^2}\left(\frac{\widehat{mc}_t + \hat{\mu}_t^p}{(1+\phi_p\bar{\mu}_p\bar{\sigma}_p)} - \hat{p}_t^p\right)$$

- Plug (38) into this:

$$E_t \hat{v}_{t+1}^p = \hat{v}_t^p + \frac{(1-\delta_p\beta)^2}{(\delta_p\beta)^2}\frac{\delta_p}{1-\delta_p}\hat{\psi}_t^p - \frac{(1-\delta_p\beta)^2}{(\delta_p\beta)^2}\frac{\delta_p}{1-\delta_p}\hat{\pi}_t^p + \frac{(1-\delta_p\beta)^2}{(\delta_p\beta)^2}\frac{\widehat{mc}_t + \hat{\mu}_t^p}{(1+\phi_p\bar{\mu}_p\bar{\sigma}_p)} \tag{40}$$

7.2 Final Equation for $\hat{\pi}_t^p$

- Deduct (35) from its once led version to get:

$$\left(E_t \hat{\pi}_{t+1}^p - \hat{\pi}_t^p\right) = \frac{1-\delta_p}{\delta_p}\left(E_t \hat{p}_{t+1}^p - \hat{p}_t^p\right) + \left(E_t \hat{\psi}_{t+1}^p - \hat{\psi}_t^p\right)$$

- To substitute for the first term on the right-hand side, use (32), which we reproduce here:

$$E_t \hat{p}_{t+1}^p - \hat{p}_t^p = \frac{-2\delta_p\beta}{(1-\delta_p\beta)}E_t \hat{v}_{t+1}^p - E_t \hat{\pi}_{t+1}^p + \frac{(1+\delta_p\beta)}{(1-\delta_p\beta)}\hat{v}_t^p$$

- To substitute for the second term on the right-hand side, use (37), which we reproduce here:

$$\left(E_t \hat{\psi}_{t+1}^p - \hat{\psi}_t^p\right) = (1-\delta_p)\hat{v}_t^p - (1-\delta_p)\,\hat{\psi}_t^p$$

- Substitute:

$$\begin{aligned}\left(E_t \hat{\pi}_{t+1}^p - \hat{\pi}_t^p\right) &= (1-\delta_p)\hat{v}_t^p - (1-\delta_p)\,\hat{\psi}_t^p \\ &+ \frac{1-\delta_p}{\delta_p}\left[\frac{-2\delta_p\beta}{(1-\delta_p\beta)}E_t \hat{v}_{t+1}^p - E_t \hat{\pi}_{t+1}^p + \frac{(1+\delta_p\beta)}{(1-\delta_p\beta)}\hat{v}_t^p\right]\end{aligned}$$

- Equivalently:

$$E_t \hat{\pi}_{t+1}^p\left(1 + \frac{1-\delta_p}{\delta_p}\right) = \hat{\pi}_t^p - \frac{1-\delta_p}{\delta_p}\frac{2\delta_p\beta}{(1-\delta_p\beta)}E_t \hat{v}_{t+1}^p + \left(\frac{1-\delta_p}{\delta_p}\frac{(1+\delta_p\beta)}{(1-\delta_p\beta)} + (1-\delta_p)\right)\hat{v}_t^p - (1-\delta_p)\,\hat{\psi}_t^p$$

- Simplify and substitute from (40) for $E_t \hat{v}^p_{t+1}$:

$$E_t \hat{\pi}^p_{t+1} \frac{1}{\delta_p} = \hat{\pi}^p_t + \left(\frac{1-\delta_p}{\delta_p} \frac{(1+\delta_p \beta)}{(1-\delta_p \beta)} + (1-\delta_p) \right) \hat{v}^p_t - (1-\delta_p) \hat{\psi}^p_t$$

$$- \frac{1-\delta_p}{\delta_p} \frac{2\delta_p \beta}{(1-\delta_p \beta)} \left[\hat{v}^p_t + \frac{(1-\delta_p \beta)^2}{(\delta_p \beta)^2} \frac{\delta_p}{1-\delta_p} \hat{\psi}^p_t - \frac{(1-\delta_p \beta)^2}{(\delta_p \beta)^2} \frac{\delta_p}{1-\delta_p} \hat{\pi}^p_t + \frac{(1-\delta_p \beta)^2}{(\delta_p \beta)^2} \frac{\widehat{mc}_t + \hat{\mu}^p_t}{(1+\phi_p \bar{\mu}_p \bar{\sigma}_p)} \right]$$

- Equivalently:

$$E_t \hat{\pi}^p_{t+1} \frac{1}{\delta_p} = \hat{\pi}^p_t \left(1 + \frac{2(1-\delta_p \beta)}{(\delta_p \beta)} \right) + \hat{v}^p_t \left(\frac{1-\delta_p}{\delta_p} \frac{(1+\delta_p \beta)}{(1-\delta_p \beta)} + (1-\delta_p) - \frac{1-\delta_p}{\delta_p} \frac{2\delta_p \beta}{(1-\delta_p \beta)} \right)$$

$$- \hat{\psi}^p_t \left(\frac{2(1-\delta_p \beta)}{(\delta_p \beta)} + (1-\delta_p) \right) - \frac{1-\delta_p}{\delta_p} \frac{2(1-\delta_p \beta)}{(\delta_p \beta)} \frac{\widehat{mc}_t + \hat{\mu}^p_t}{(1+\phi_p \bar{\mu}_p \bar{\sigma}_p)}$$

- Equivalently:

$$E_t \hat{\pi}^p_{t+1} \frac{1}{\delta_p} = \hat{\pi}^p_t \left(\frac{\delta_p \beta + 2 - 2\delta_p \beta}{\delta_p \beta} \right) + \hat{v}^p_t \left(\frac{1-\delta_p}{\delta_p} \frac{(1+\delta_p \beta - 2\delta_p \beta)}{(1-\delta_p \beta)} + (1-\delta_p) \right)$$

$$- \hat{\psi}^p_t \left(\frac{2 - 2\delta_p \beta + \delta_p \beta - \delta_p^2 \beta}{\delta_p \beta} \right) - \frac{1-\delta_p}{\delta_p} \frac{2(1-\delta_p \beta)}{(\delta_p \beta)} \frac{\widehat{mc}_t + \hat{\mu}^p_t}{(1+\phi_p \bar{\mu}_p \bar{\sigma}_p)}$$

- Equivalently:

$$E_t \hat{\pi}^p_{t+1} \frac{1}{\delta_p} = \hat{\pi}^p_t \left(\frac{2 - \delta_p \beta}{\delta_p \beta} \right) + \hat{v}^p_t \left(\frac{(1-\delta_p)(1+\delta_p)}{\delta_p} \right)$$

$$- \hat{\psi}^p_t \left(\frac{2 - \delta_p \beta - \delta_p^2 \beta}{\delta_p \beta} \right) - \frac{1-\delta_p}{\delta_p} \frac{2(1-\delta_p \beta)}{(\delta_p \beta)} \frac{\widehat{mc}_t + \hat{\mu}^p_t}{(1+\phi_p \bar{\mu}_p \bar{\sigma}_p)}$$

- Equivalently:

$$E_t \hat{\pi}^p_{t+1} = \hat{\pi}^p_t \left(\frac{2}{\beta} - \delta_p \right) + \hat{v}^p_t \left((1-\delta_p)(1+\delta_p) \right) \tag{41}$$

$$+ \hat{\psi}^p_t \left(\delta_p (1+\delta_p) - \frac{2}{\beta} \right) - \frac{2(1-\delta_p)(1-\delta_p \beta)}{(\delta_p \beta)(1+\phi_p \bar{\mu}_p \bar{\sigma}_p)} (\widehat{mc}_t + \hat{\mu}^p_t)$$

8 THE CALVO-YUN CASE

8.1 Profit Maximization

- Pricing policy of firm j that reoptimizes at t, choosing V_t^p and indexing to π_t^* (the current inflation target):

$$P_{t+k}(j) = V_t^p \Pi_{t,k}^*$$

- Define additional terms:
  - Cumulative inflation targets:

$$\Pi_{t,k}^* \equiv \prod_{j=1}^{k} \pi_{t+j}^* \text{ for } k \geq 1 \;\; (\equiv 1 \text{ for } k = 0)$$

  - Cumulative lagged aggregate inflation deviation:

$$\hat{\Pi}_{t,k}^* \equiv \sum_{j=1}^{k} \hat{\pi}_{t+j}^* \text{ for } k \geq 1 \;\; (\equiv 0 \text{ for } k = 0)$$

- Profit maximization:

$$\underset{V_t^p}{Max} \quad E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \lambda_{t+k} \left[\frac{V_t^p \Pi_{t,k}^*}{P_t \Pi_{t,k}} y_{t+k}(j) - mc_{t+k} y_{t+k}(j) - \frac{\phi_p}{2} \frac{(y_{t+k}(j) - Y_{t+k})^2}{Y_{t+k}} \right] \quad , \; s.t.$$

$$y_{t+k}(j) = Y_{t+k} \left(\frac{V_t^p \Pi_{t,k}^*}{P_t \Pi_{t,k}} \right)^{-\sigma_{t+k}^p}$$

- Substitute constraints:

$$\underset{V_t^p, v_t^p}{Max} E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \lambda_{t+k} \left[\left(\frac{V_t^p \Pi_{t,k}^*}{P_t \Pi_{t,k}} \right)^{1-\sigma_{t+k}^p} Y_{t+k} - mc_{t+k} \left(\frac{V_t^p \Pi_{t,k}^*}{P_t \Pi_{t,k}} \right)^{-\sigma_{t+k}^p} Y_{t+k} - \frac{\phi_p}{2} \frac{(y_{t+k}(j) - Y_{t+k})^2}{Y_{t+k}} \right]$$

- FOC for V_t^p, normalized by inflation target:

$$p_t = \frac{E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \check{\lambda}_{t+k} \check{y}_{t+k}(j) \sigma_{t+k}^p \left(mc_{t+k} + \phi_p \left(\frac{\check{y}_{t+k}(j) - \check{Y}_{t+k}}{\check{Y}_{t+k}} \right) \right)}{E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \check{\lambda}_{t+k} \check{y}_{t+k}(j) (\sigma_{t+k}^p - 1) \left(\frac{1}{\check{\Pi}_{t,k}} \right)} \qquad (42)$$

- Linearization of goods demand:

$$\left(\hat{y}_{t+k}(j) - \hat{Y}_{t+k} \right) = -\bar{\sigma}_p \left(\hat{p}_t^p - \hat{\Pi}_{t,k}^p \right)$$

- Rewrite (42):

$$
\begin{aligned}
& p_t E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \check{\lambda}_{t+k} \check{y}_{t+k}(j) (\sigma^p_{t+k} - 1) \left(\frac{1}{\check{\Pi}_{t,k}} \right) \\
= \; & E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \check{\lambda}_{t+k} \check{y}_{t+k}(j) \sigma^p_{t+k} \left(mc_{t+k} + \phi_p \left(\frac{\check{y}_{t+k}(j) - \check{Y}_{t+k}}{\check{Y}_{t+k}} \right) \right)
\end{aligned}
$$

- Linearization of (42):

$$
\frac{\hat{p}^p_t}{1 - \delta_p \beta} - E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \hat{\Pi}^p_{t,k}
$$

$$
= E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \left[\widehat{mc}_{t+k} + \hat{\mu}^p_{t+k} + \phi_p \bar{\mu}_p \left(\hat{y}_{t+k}(j) - \hat{Y}_{t+k} \right) \right]
$$

- Equivalently:

$$
\frac{\hat{p}^p_t}{1 - \delta_p \beta} - E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \hat{\Pi}^p_{t,k} = E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \frac{\left(\widehat{mc}_{t+k} + \hat{\mu}^p_{t+k} \right)}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)}
$$

- Equivalently:

$$
\frac{\hat{p}^p_t}{1 - \delta_p \beta} = E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \left[\frac{\left(\widehat{mc}_{t+k} + \hat{\mu}^p_{t+k} \right)}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} + \hat{\Pi}^p_{t,k} \right]
$$

- Quasi-Differencing:

$$
\hat{p}^p_t - \delta_p \beta E_t \hat{p}^p_{t+1} = (1 - \delta_p \beta) \left[\frac{\widehat{mc}_t + \hat{\mu}^p_t}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} \right] + \delta_p \beta E_t \hat{\pi}^p_{t+1} \tag{43}
$$

8.2 The Price Index

- Price Level:

$$
P_t = \left[(1 - \delta_p) \sum_{s=0}^{\infty} \delta_p^s \left[V^p_{t-s} \Pi^*_{t-s,s} \right]^{1 - \sigma^p_t} \right]^{\frac{1}{1 - \sigma^p_t}}
$$

- Deflate by current target price level P^*_t and write out terms ($\check{P}_t = P_t / P^*_t$, $\check{V}^p_t = V^p_t / P^*_t$):

$$
\left(\check{P}_t \right)^{1-\sigma^p_t} = (1 - \delta_p) \left(\check{V}^p_t \right)^{1-\sigma^p_t} + (1 - \delta_p) \delta_p \left(\check{V}^p_{t-1} \right)^{1-\sigma^p_t} + (1 - \delta_p) \delta_p^2 \left(\check{V}^p_{t-2} \right)^{1-\sigma^p_t} + (1 - \delta_p) \delta_p^3 \left(\check{V}^p_{t-3} \right)^{1-\sigma^p_t} + \ldots
$$

- Divide by $\check{P}_{t-1}$:

$$\begin{aligned}
\left(\frac{\check{P}_t}{\check{P}_{t-1}}\right)^{1-\sigma_t^p} &= (1-\delta_p)\left(\frac{\check{V}_t^p}{\check{P}_t}\right)^{1-\sigma_t^p}\left(\frac{\check{P}_t}{\check{P}_{t-1}}\right)^{1-\sigma_t^p} \\
&+(1-\delta_p)\delta_p\left(\frac{\check{V}_{t-1}^p}{\check{P}_{t-1}}\right)^{1-\sigma_t^p} \\
&+(1-\delta_p)\delta_p^2\left(\frac{\check{V}_{t-2}^p}{\check{P}_{t-2}}\right)^{1-\sigma_t^p}\left(\frac{\check{P}_{t-2}}{\check{P}_{t-1}}\right)^{1-\sigma_t^p} \\
&+(1-\delta_p)\delta_p^3\left(\frac{\check{V}_{t-3}^p}{\check{P}_{t-3}}\right)^{1-\sigma_t^p}\left(\frac{\check{P}_{t-3}}{\check{P}_{t-2}}\right)^{1-\sigma_t^p}\left(\frac{\check{P}_{t-2}}{\check{P}_{t-1}}\right)^{1-\sigma_t^p}+...
\end{aligned}$$

- Use $\check{P}_t/\check{P}_{t-1} = \check{\pi}_t^p = \frac{\pi_t^p}{\pi_t^*}$:

$$\begin{aligned}
\left(\check{\pi}_t^p\right)^{1-\sigma_t^p} &= (1-\delta_p)\left(p_t\right)^{1-\sigma_t^p}\left(\check{\pi}_t^p\right)^{1-\sigma_t^p} \\
&+(1-\delta_p)\delta_p\left(p_{t-1}\right)^{1-\sigma_t^p} \\
&+(1-\delta_p)\delta_p^2\left(p_{t-2}\right)^{1-\sigma_t^p}\left(\frac{1}{\check{\pi}_{t-1}^p}\right)^{1-\sigma_t^p} \\
&+(1-\delta_p)\delta_p^3\left(p_{t-3}\right)^{1-\sigma_t^p}\left(\frac{1}{\check{\pi}_{t-2}^p}\right)^{1-\sigma_t^p}\left(\frac{1}{\check{\pi}_{t-1}^p}\right)^{1-\sigma_t^p}+...
\end{aligned}$$

- Divide through by $\left(\check{\pi}_t^p\right)^{1-\sigma_t^p}$:

$$\begin{aligned}
1 &= (1-\delta_p)p_t^{1-\sigma_t^p} \\
&+(1-\delta_p)\delta_p p_{t-1}^{1-\sigma_t^p}\left(\frac{1}{\check{\pi}_t^p}\right)^{1-\sigma_t^p} \\
&+(1-\delta_p)\delta_p^2 p_{t-2}^{1-\sigma_t^p}\left(\frac{1}{\check{\pi}_{t-1}^p\check{\pi}_t^p}\right)^{1-\sigma_t^p} \\
&+(1-\delta_p)\delta_p^3 p_{t-3}^{1-\sigma_t^p}\left(\frac{1}{\check{\pi}_{t-2}^p\check{\pi}_{t-1}^p\check{\pi}_t^p}\right)^{1-\sigma_t^p}+...
\end{aligned}$$

- Linearize:

$$0 = \hat{p}_t^p + \delta_p\left(\hat{p}_{t-1}^p - \hat{\pi}_t^p\right) + \delta_p^2\left(\hat{p}_{t-2}^p - \hat{\pi}_{t-1}^p - \hat{\pi}_t^p\right) + \delta_p^3\left(\hat{p}_{t-3}^p - \hat{\pi}_{t-2}^p - \hat{\pi}_{t-1}^p - \hat{\pi}_t^p\right) + ...$$

- Cancel terms and bring $\hat{\pi}_t^p$ onto left-hand side:

$$\hat{\pi}_t^p\left(\delta_p + \delta_p^2 + \delta_p^3 + ...\right) = \hat{p}_t^p + \delta_p\hat{p}_{t-1}^p + \delta_p^2\left(\hat{p}_{t-2}^p - \hat{\pi}_{t-1}^p\right) + \delta_p^3\left(\hat{p}_{t-3}^p - \hat{\pi}_{t-2}^p - \hat{\pi}_{t-1}^p\right) + ...$$

- Equivalently:

$$\hat{\pi}_t^p = \frac{1-\delta_p}{\delta_p}\hat{p}_t^p + (1-\delta_p)\hat{p}_{t-1}^p + (1-\delta_p)\delta_p\left(\hat{p}_{t-2}^p - \hat{\pi}_{t-1}^p\right) + (1-\delta_p)\delta_p^2\left(\hat{p}_{t-3}^p - \hat{\pi}_{t-2}^p - \hat{\pi}_{t-1}^p\right) + ...$$

- Lag the last equation and multiply by δ_p:

$$\delta_p\hat{\pi}_{t-1}^p = (1-\delta_p)\hat{p}_{t-1}^p + (1-\delta_p)\delta_p\hat{p}_{t-2}^p + (1-\delta_p)\delta_p^2\left(\hat{p}_{t-3}^p - \hat{\pi}_{t-2}^p\right) +$$

- Deduct the last equation from the preceding one:

$$\hat{\pi}_t^p = \frac{1-\delta_p}{\delta_p}\hat{p}_t^p \tag{44}$$

8.3 Final Inflation Dynamics

- Plug (44) into (43):

$$\frac{\delta_p}{1-\delta_p}\hat{\pi}_t^p - \delta_p\beta\frac{\delta_p}{1-\delta_p}E_t\hat{\pi}_{t+1}^p = (1-\delta_p\beta)\left[\frac{\widehat{mc}_t + \hat{\mu}_t^p}{(1+\phi_p\bar{\mu}_p\bar{\sigma}_p)}\right] + \delta_p\beta E_t\hat{\pi}_{t+1}^p$$

- Simplify:

$$\hat{\pi}_t^p = \beta E_t\hat{\pi}_{t+1}^p + \frac{(1-\delta_p\beta)(1-\delta_p)}{\delta_p}\left[\frac{\widehat{mc}_t + \hat{\mu}_t^p}{(1+\phi_p\bar{\mu}_p\bar{\sigma}_p)}\right] \tag{45}$$

- This is the New Keynesian Phillips Curve for the Calvo-Yun case, which replaces the three equation pricing block of the optimal indexation system.

9 HOUSEHOLD WAGE SETTING

9.1 Utility Maximization

- Wage setting policy of a worker i that reoptimizes at t, choosing V_t^w and v_t^w (a gross wage inflation rate):

$$W_{t+k}(i) = V_t^w \left(v_t^w\right)^k$$

- Define terms:
  - Front-loading term:

$$p_t^w \equiv \frac{V_t^w}{W_t}$$

  - Real wage:

$$w_t = \frac{W_t}{P_t}$$

  - Wage inflation rates re-scaled by the inflation target, with $\breve{\pi}_t^w = \pi_t^w / \pi_t^*$.
  - Cumulative aggregate rescaled wage inflation:

$$\breve{\Pi}_{t,k}^w = \Pi_{j=1}^k \breve{\pi}_{t+j}^w \text{ for } k \geq 1 \;\; (\equiv 1 \text{ for } k = 0)$$

  - Cumulative aggregate rescaled wage inflation deviation:

$$\hat{\Pi}_{t,k}^w = \Sigma_{j=1}^k \hat{\pi}_{t+j}^w \text{ for } k \geq 1 \;\; (\equiv 0 \text{ for } k = 0)$$

- Utility maximization - relevant part of the problem:

$$\underset{V_t^w, v_t^w}{Max} \quad E_t \sum_{k=0}^{\infty} \left(\delta_w \beta\right)^k \left[-\psi S_t^L \frac{\left(L_{t+k}(i)\right)^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} + \lambda_{t+k} \frac{V_t^w \left(v_t^w\right)^k}{W_{t+k}} \frac{W_{t+k}}{P_{t+k}} L_{t+k}(i) \right.$$
$$\left. - \lambda_{t+k} \frac{\phi_w}{2} \frac{W_{t+k}}{P_{t+k}} \frac{\left(L_{t+k}(i) - \ell_{t+k}\right)^2}{\ell_{t+k}} \right] \quad , \; s.t.$$
$$L_{t+k}(i) = \ell_{t+k} \left(\frac{V_t^w \left(v_t^w\right)^k}{W_{t+k}} \right)^{-\sigma_t^w}$$

9.2 First-Order Conditions

- Real wage rescaled by technology, with $\check{w}_t = w_t / S_t^y$.
- Firm specific wage inflation rescaled by the inflation target $\check{v}_t^w = v_t^w / \pi_t^*$.
  - By the unit root property we can ignore future changes in the inflation target (see discussion above for firms):

$$E_t \hat{v}_{t+k}^w = E_t \left(\ln \left(v_{t+k}^w \right) - \ln(\pi_{t+k}^*) \right) = E_t \left(\ln \left(v_{t+k}^w \right) - \ln(\pi_t^*) \right)$$

- FOC for V_t^w:

$$\begin{aligned} & E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k \check{\lambda}_{t+k} L_{t+k}(i) \left(\check{w}_{t+k} \left(\sigma_{t+k}^w - 1 \right) p_t^w \frac{(\check{v}_t^w)^k}{\check{\Pi}_{t,k}^w} - \phi_w \check{w}_{t+k} \sigma_{t+k}^w \left(\frac{L_{t+k}(i) - \ell_{t+k}}{\ell_{t+k}} \right) \right) \\ = \; & E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k \psi \sigma_{t+k}^w S_{t+k}^L \left(L_{t+k}(i) \right)^{1+\frac{1}{\eta}} \end{aligned} \tag{46}$$

- FOC w.r.t. v_t^w:

$$\begin{aligned} & E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k k \check{\lambda}_{t+k} L_{t+k}(i) \left(\check{w}_{t+k} \left(\sigma_{t+k}^w - 1 \right) p_t^w \frac{(\check{v}_t^w)^k}{\check{\Pi}_{t,k}^w} - \phi_w \check{w}_{t+k} \sigma_{t+k}^w \left(\frac{L_{t+k}(i) - \ell_{t+k}}{\ell_{t+k}} \right) \right) \\ = \; & E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k k \psi \sigma_{t+k}^w S_{t+k}^L \left(L_{t+k}(i) \right)^{1+\frac{1}{\eta}} \end{aligned} \tag{47}$$

9.3 Linearization

9.3.1 Linearization for V_t^w

- First step:

$$E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k \left[\hat{\lambda}_{t+k} + \hat{L}_{t+k}(i) + \hat{w}_{t+k} + \left(\hat{p}_t^w + k\hat{v}_t^w - \hat{\Pi}_{t,k}^w\right)\right]$$

$$= E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k \left[\hat{\mu}_{t+k}^w + \hat{S}_{t+k}^L + \left(1 + \frac{1}{\eta}\right) \hat{L}_{t+k}(i) + \phi_w \bar{\mu}_w \left(\hat{L}_{t+k}(i) - \hat{\ell}_{t+k}\right)\right]$$

- Linearization of labor demand:

$$\left(\hat{L}_{t+k}(i) - \hat{\ell}_{t+k}\right) = -\bar{\sigma}_w \left(\hat{p}_t^w + k\hat{v}_t^w - \hat{\Pi}_{t,k}^w\right)$$

- Marginal rate of substitution:
  - MRS in levels:

$$mrs_t = \frac{S_t^L \psi L_t(i)^{\frac{1}{\eta}}}{\lambda_t}$$

  - Log-linearized:

$$\widehat{mrs}_t = \frac{1}{\eta} \hat{L}_t(i) + \hat{S}_t^L - \hat{\lambda}_t$$

  - Combine with the expression for labor demand (note that, for contemporaneous terms, $k = 0$):

$$\widehat{mrs}_t = \frac{1}{\eta} \hat{\ell}_t - \frac{\bar{\sigma}_w}{\eta} \hat{p}_t^w + \hat{S}_t^L - \hat{\lambda}_t$$

- Combine the above:

$$E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k \left(\hat{p}_t^w + k\hat{v}_t^w - \hat{\Pi}_{t,k}^w\right) = E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k \frac{\left(\widehat{mrs}_{t+k} - \hat{w}_{t+k} + \hat{\mu}_{t+k}^w\right)}{(1 + \phi_w \bar{\mu}_w \bar{\sigma}_w)}$$

- Apply formulas (21) and (22):

$$\frac{\hat{p}_t^w}{1 - \delta_w \beta} + \frac{\hat{v}_t^w \delta_w \beta}{(1 - \delta_w \beta)^2} = E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k \left[\frac{\left(\widehat{mrs}_{t+k} - \hat{w}_{t+k} + \hat{\mu}_{t+k}^w\right)}{(1 + \phi_w \bar{\mu}_w \bar{\sigma}_w)} + \hat{\Pi}_{t,k}^w\right] \qquad (48)$$

- With the appropriate change in notation this is exactly identical to equation (25) for price setting, after replacing $\left(\widehat{mc}_{t+k} + \hat{\mu}_{t+k}\right) / \left(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p\right)$ with $\left(\widehat{mrs}_{t+k} - \hat{w}_{t+k} + \hat{\mu}_{t+k}^w\right) / (1 + \phi_w \bar{\mu}_w \bar{\sigma}_w)$.

9.3.2 Linearization for v_t^w

- First step:

$$E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k k \left[\hat{\lambda}_{t+k} + \hat{L}_{t+k}(i) + \hat{w}_{t+k} + \left(\hat{p}_t^w + k\hat{v}_t^w - \hat{\Pi}_{t,k}^w \right) \right]$$

$$= E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k k \left[\hat{\mu}_{t+k}^w + \hat{S}_{t+k}^L + \left(1 + \frac{1}{\eta} \right) \hat{L}_{t+k}(i) + \phi_w \bar{\mu}_w \left(\hat{L}_{t+k}(i) - \hat{\ell}_{t+k} \right) \right]$$

- Combine with the above:

$$E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k k \left(\hat{p}_t^w + k\hat{v}_t^w - \hat{\Pi}_{t,k}^w \right) = E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k k \frac{\left(\widehat{mrs}_{t+k} - \hat{w}_{t+k} + \hat{\mu}_{t+k}^w \right)}{\left(1 + \phi_w \bar{\mu}_w \bar{\sigma}_w \right)}$$

- Apply formulas (21) and (22):

$$\begin{aligned} & \frac{\hat{p}_t^w \delta_w \beta}{(1 - \delta_w \beta)^2} + \frac{\hat{v}_t^w \delta_w \beta (1 + \delta_w \beta)}{(1 - \delta_w \beta)^3} \\ = \quad & E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k k \left[\frac{\left(\widehat{mrs}_{t+k} - \hat{w}_{t+k} + \hat{\mu}_{t+k}^w \right)}{\left(1 + \phi_w \bar{\mu}_w \bar{\sigma}_w \right)} + \hat{\Pi}_{t,k}^w \right] \end{aligned} \tag{49}$$

- With the appropriate change in notation this is exactly identical to equation (26) for price setting, after replacing $\left(\widehat{mc}_{t+k} + \hat{\mu}_{t+k}^p \right) / \left(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p \right)$ with $\left(\widehat{mrs}_{t+k} - \hat{w}_{t+k} + \hat{\mu}_{t+k}^w \right) /$ $\left(1 + \phi_w \bar{\mu}_w \bar{\sigma}_w \right)$.

9.4 Final Wage Inflation Dynamics

- Given the identical forms of (48)/(25) and (49)/(26), we obtain after quasi-differencing as the end result an equation analogous to (33):

$$\left(E_t\hat{v}^w_{t+1} - \hat{v}^w_t\right) = \frac{(1-\delta_w\beta)^2}{(\delta_w\beta)^2}\left(\frac{(\widehat{mrs}_t - \hat{w}_t + \hat{\mu}^w_t)}{(1+\phi_w\bar{\mu}_w\bar{\sigma}_w)} - \hat{p}^w_t\right) \tag{50}$$

- Furthermore, the derivation of the wage index is identical to that for the price index above. The correct expression for wage inflation is:

$$\hat{\pi}^w_t = \hat{w}_t - \hat{w}_{t-1} + \hat{g}_t - \hat{\pi}^p_t \tag{51}$$

- Furthermore, the derivation of the wage index is identical to that for the price index above. We get:

$$\hat{\pi}^w_t = \frac{1-\delta_w}{\delta_w}\hat{p}^w_t + \hat{\psi}^w_t \tag{52}$$

$$\hat{\psi}^w_t = \delta_w\hat{\psi}^w_{t-1} + (1-\delta_w)\hat{v}^w_{t-1} - \hat{\varepsilon}^{\pi^*}_t \tag{53}$$

- Combining this with the results of quasi-differencing we obtain the equations for $E_t\hat{v}^w_{t+1}$ and $E_t\hat{\pi}^w_{t+1}$:

$$\begin{aligned} E_t\hat{v}^w_{t+1} &= \hat{v}^w_t + \frac{(1-\delta_w\beta)^2}{(\delta_w\beta)^2}\frac{\delta_w}{1-\delta_w}\hat{\psi}^w_t - \frac{(1-\delta_w\beta)^2}{(\delta_w\beta)^2}\frac{\delta_w}{1-\delta_w}\hat{\pi}^w_t \\ &\quad + \frac{(1-\delta_w\beta)^2}{(\delta_w\beta)^2}\frac{(\widehat{mrs}_t - \hat{w}_t + \hat{\mu}^w_t)}{(1+\phi_w\bar{\mu}_w\bar{\sigma}_w)} \end{aligned} \tag{54}$$

$$\begin{aligned} E_t\hat{\pi}^w_{t+1} &= \hat{\pi}^w_t\left(\frac{2}{\beta} - \delta_w\right) + \hat{v}^w_t\left((1-\delta_w)(1+\delta_w)\right) \\ &\quad + \hat{\psi}^w_t\left(\delta_w(1+\delta_w) - \frac{2}{\beta}\right) - \frac{2(1-\delta_w)(1-\delta_w\beta)}{(\delta_w\beta)}\frac{(\widehat{mrs}_t - \hat{w}_t + \hat{\mu}^w_t)}{(1+\phi_w\bar{\mu}_w\bar{\sigma}_w)} \end{aligned} \tag{55}$$

- We can also use (52) to get a final expression for the marginal rate of substitution:

$$\widehat{mrs}_t = \frac{1}{\eta}\hat{\ell}_t - \frac{\overline{\sigma^w}}{\eta}\frac{\delta_w}{1-\delta_w}\hat{\pi}^w_t + \frac{\overline{\sigma^w}}{\eta}\frac{\delta_w}{1-\delta_w}\hat{\psi}^w_t + \hat{S}^L_t - \hat{\lambda}_t \tag{56}$$

9.5 The Calvo-Yun Case

By analogy with the derivation under price setting we obtain the following for wage setting:

$$\hat{\pi}^w_t = \beta E_t\hat{\pi}^w_{t+1} + \frac{(1-\delta_w\beta)(1-\delta_w)}{\delta_w}\left(\frac{\widehat{mrs}_t - \hat{w}_t + \hat{\mu}^w_t}{(1+\phi_w\bar{\mu}_w\bar{\sigma}_w)}\right) \tag{57}$$

10 GOVERNMENT AND MARKET CLEARING

- Interest rate rule in levels:
  - Re-scaled by dividing through by π_t^*.
  - Remember that $\check{i}_t = i_t/\pi_t^*$.
  - Let $\bar{r} = \bar{g}/\beta$.

$$\left(\frac{i_t}{\pi_t^*}\right)^4 = \left[\left(\frac{i_{t-1}}{\pi_{t-1}^*}\frac{\pi_{t-1}^*}{\pi_t^*}\right)^4\right]^{\theta^{int}} \left[\bar{r}^4 \frac{\pi_{t+3}^p \pi_{t+2}^p \pi_{t+1}^p \pi_t^p}{(\pi_t^*)^4}\right]^{1-\theta^{int}+\theta^\pi} \left[\frac{\check{Y}_{t+3}}{\check{Y}_{t-1}}\right]^{\theta^y}$$

  - Equivalently:

$$\left(\check{i}_t\right)^4 = \left[\left(\frac{\check{i}_{t-1}}{\varepsilon_t^{\pi^*}}\right)^4\right]^{\theta^{int}} \left[\bar{r}^4 \check{\pi}_{t+3}^p \check{\pi}_{t+2}^p \check{\pi}_{t+1}^p \check{\pi}_t^p\right]^{1-\theta^{int}+\theta^\pi} \left[\frac{\check{Y}_{t+3}}{\check{Y}_{t-1}}\right]^{\theta^y}$$

- Linearized interest rate rule:

$$\begin{aligned} \hat{i}_t &= \theta^{int}\left(\hat{i}_{t-1} - \hat{\varepsilon}_t^{\pi^*}\right) + (1-\theta^{int}+\theta^\pi)E_t\left(\frac{\hat{\pi}_{t+3}^p + \hat{\pi}_{t+2}^p + \hat{\pi}_{t+1}^p + \hat{\pi}_t^p}{4}\right) \\ &\quad +\theta^y E_t\left(\frac{\hat{Y}_{t+3} - \hat{Y}_{t-1}}{4}\right) + \frac{\hat{S}_t^{int}}{4} \end{aligned} \tag{58}$$

- Real Interest Rate:

$$\hat{r}_t = \hat{i}_t - E_t\hat{\pi}_{t+1}^p \tag{59}$$

- Steady State Spending = Fixed Fraction of GDP:

$$\begin{aligned} \overline{GOV} &= s_g\bar{Y} \\ G\check{O}V_t &= S_t^{gov}\overline{GOV} \end{aligned}$$

- Goods Market Clearing:
  - In levels:

$$Y_t = C_t + I_t + GOV_t$$

  - Rescaled by technology:

$$\check{Y}_t = \check{C}_t + \check{I}_t + G\check{O}V_t$$

  - Linearization:

$$\bar{Y}\hat{Y}_t = \bar{C}\hat{C}_t + \bar{I}\hat{I}_t + \overline{GOV}\hat{S}_t^g \tag{60}$$

11 STATIONARY EXOGENOUS SHOCKS

- Firm Risk:

$$\hat{S}_t^{\sigma} = \rho^{\sigma} \hat{S}_{t-1}^{\sigma} + \hat{\varepsilon}_t^{\sigma} \tag{61}$$

- Consumption:

$$\hat{S}_t^{c} = \rho^{c} \hat{S}_{t-1}^{c} + \hat{\varepsilon}_t^{c} \tag{62}$$

- Labor Supply:

$$\hat{S}_t^{L} = \rho^{L} \hat{S}_{t-1}^{L} + \hat{\varepsilon}_t^{L} \tag{63}$$

- Investment Demand:

$$\hat{S}_t^{inv} = \rho^{inv} \hat{S}_{t-1}^{inv} + \hat{\varepsilon}_t^{inv} \tag{64}$$

- Government Spending:

$$\hat{S}_t^{gov} = \rho^{gov} \hat{S}_{t-1}^{gov} + \hat{\varepsilon}_t^{gov} \tag{65}$$

- Monetary Policy:

$$\hat{S}_t^{int} = \rho^{int} \hat{S}_{t-1}^{int} + \hat{\varepsilon}_t^{int} \tag{66}$$

- Goods Mark-Up:

$$\hat{\mu}_t^{p} = \hat{\varepsilon}^{\mu^{p}} \tag{67}$$

- Labor Mark-Up:

$$\hat{\mu}_t^{w} = \hat{\varepsilon}^{\mu^{w}} \tag{68}$$

12 COMPLETE DYNAMIC SYSTEM

12.1 Steady State and Parameter Calibration

12.1.1 Calibrated Directly

$$v = 0.7 \text{ (Habit Persistence)}$$
$$\eta = 0.5 \text{ (Labor Supply Elasticity)}$$

$$\phi_I = 0.7 \text{ (Investment Adjustment Costs)}$$
$$\sigma_a = 15 \text{ (Capacity Utilization Elasticity)}$$

$$\delta_p = 0.75 \text{ (Calvo delta for Prices)}$$
$$\delta_w = 0.75 \text{ (Calvo delta for Wages)}$$

$$\bar{\mu}_p = 1.23 \text{ (Price Markup)} \tag{SS2}$$

$$\bar{\mu}_w = 1.16 \text{ (Wage Markup)} \tag{SS2}$$

$$\phi_p = 0.75 \text{ (Deviation Cost for Prices)}$$
$$\phi_w = 0.75 \text{ (Deviation Cost for Wages)}$$
$$\theta^{int} = 0.75 \text{ (Interest Rate Smoothing)}$$
$$\theta^{\pi} = 0.25 \text{ (One minus Inflation Coefficient)}$$

$$\bar{L} = 1/3 \quad \text{(Labor Supply, } \psi \text{ is endogenous to that choice)} \tag{SS1}$$

$$s_L = 0.64 \text{ (Labor Share - after adjusting for markups)} \tag{SS3}$$
$$s_G = 0.18$$

$$\beta = 0.99 \tag{SS4}$$

$$\bar{g} = 1.005 \tag{SS5}$$

$$\bar{q} = 1 \tag{SS6}$$

$$\Delta = 0.025$$

$$\overline{\omega} = 0.479 \text{ (preliminary, from GIMF)} \tag{SS7}$$

$$\bar{\sigma} = 0.452 \text{ (preliminary, from GIMF)} \tag{SS8}$$

$$\bar{\xi} = 0.357 \text{ (preliminary, from GIMF)} \tag{SS9}$$

12.1.2 Derived

$$\bar{r} = \frac{\bar{g}}{\beta} \tag{SS10}$$

$$\alpha = 1 - \bar{\mu}_p s_L \tag{SS11}$$

$$\overline{mc} = \frac{1}{\bar{\mu}_p} \tag{SS12}$$

$$\bar{z} = \frac{\ln(\overline{\omega}) + \frac{1}{2}\bar{\sigma}^2}{\bar{\sigma}} \tag{SS13}$$

$$\bar{\Gamma} = \frac{1}{2}erfc\left(\frac{\bar{\sigma} - \bar{z}}{\sqrt{2}}\right) + \overline{\omega}\left(1 - \frac{1}{2}erfc\left(\frac{-\bar{z}}{\sqrt{2}}\right)\right) \tag{SS14}$$

$$\bar{G} = \frac{1}{2}erfc\left(\frac{\bar{\sigma} - \bar{z}}{\sqrt{2}}\right) \tag{SS15}$$

$$f\left(\overline{\omega}\right) = \frac{1}{\sqrt{2\pi}\overline{\omega}\bar{\sigma}} \exp\left\{-\frac{1}{2}\bar{z}^2\right\} \tag{SS16}$$

$$\bar{\Gamma}^{\omega} = 1 - \frac{1}{2}erfc\left(\frac{-\bar{z}}{\sqrt{2}}\right) \tag{SS17}$$

$$\bar{G}^{\omega} = \overline{\omega} f\left(\overline{\omega}\right) \tag{SS18}$$

$$\bar{\bar{\lambda}} = \frac{\bar{\Gamma}^{\omega}}{\bar{\Gamma}^{\omega} - \bar{\xi}\bar{G}^{\omega}} \tag{SS19}$$

$$\overline{ret}_k = \frac{\bar{r}\bar{\bar{\lambda}}}{1 - \bar{\Gamma} + \bar{\bar{\lambda}}\left(\bar{\Gamma} - \bar{\xi}\bar{G}\right)} \tag{SS20}$$

$$\bar{r}_k = \overline{ret}_k - 1 + \Delta \tag{SS21}$$

$$\bar{\Gamma}^{\omega\omega} = -f\left(\overline{\omega}\right) \tag{SS22}$$

$$\bar{\Gamma}^{\omega\sigma} = f\left(\overline{\omega}\right)\overline{\omega}\left(\bar{z} - \bar{\sigma}\right) \tag{SS23}$$

$$\bar{\Gamma}^{\sigma} = -\overline{\omega}^{2}\bar{\sigma}f\left(\overline{\omega}\right) \tag{SS24}$$

$$\bar{G}^{\omega\omega} = -f\left(\overline{\omega}\right)\frac{\bar{z}}{\bar{\sigma}} \tag{SS25}$$

$$\bar{G}^{\omega\sigma} = \frac{\overline{\omega}}{\bar{\sigma}}f\left(\overline{\omega}\right)\left[\bar{z}\left(\bar{z} - \bar{\sigma}\right) - 1\right] \tag{SS26}$$

$$\bar{G}^{\sigma} = -\overline{\omega}^{2}f\left(\overline{\omega}\right)\bar{z} \tag{SS27}$$

$$A = \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)} \tag{SS28}$$

$$\bar{w} = \left(\frac{\overline{mc}}{A\left(\bar{r}_k\right)^{\alpha}}\right)^{\frac{1}{1-\alpha}} \tag{SS29}$$

$$\bar{Y} = \frac{\bar{w}\bar{L}}{(1-\alpha)\overline{mc}} \tag{SS30}$$

$$\bar{k} = \frac{\alpha\overline{mc}\bar{Y}}{\bar{r}_k} \tag{SS31}$$

$$\bar{K} = \bar{k}\bar{g} \tag{SS32}$$

$$\bar{I} = \frac{\left(\bar{g} + \Delta - 1\right)}{\bar{g}}\bar{K} \tag{SS33}$$

$$\overline{GOV} = s_g\bar{Y} \tag{SS34}$$

$$\bar{C} = \bar{Y} - \bar{I} - \overline{GOV} \tag{SS35}$$

$$\bar{n} = \bar{K}\left(1 - \frac{\overline{ret}_k}{\bar{r}_k}\left(\bar{\Gamma} - \bar{\xi}\bar{G}\right)\right) \tag{SS36}$$

$$d = \frac{\bar{r}}{\bar{g}} - 1 + \frac{\bar{K}}{\bar{g}\bar{n}}\left(\overline{ret}_k\left(1 - \bar{\xi}\bar{G}\right) - \bar{r}\right) \tag{SS37}$$

12.2 Demand Block

$$\hat{\lambda}_t = \hat{\imath}_t + E_t\left(\hat{\lambda}_{t+1} - \hat{\pi}^p_{t+1} - \hat{g}_{t+1}\right) \tag{D1}$$

$$\hat{S}^c_t - \hat{H}_t = \hat{\lambda}_t \tag{D2}$$

$$\hat{H}_t = \frac{1}{1-\frac{\nu}{\bar{g}}}\hat{C}_t - \frac{\frac{\nu}{\bar{g}}}{1-\frac{\nu}{\bar{g}}}\left(\hat{C}_{t-1} - \hat{g}_t\right) \tag{D3}$$

$$\bar{K}\hat{K}_t = (1-\Delta)\frac{\bar{K}}{\bar{g}}\left(\hat{K}_{t-1} - \hat{g}_t\right) + \bar{I}\hat{I}_{t-1} \tag{D4}$$

$$\hat{q}_t = \phi_I\left(\hat{I}_t - \hat{I}_{t-1}\right) - \beta\phi_I\left(E_t\hat{I}_{t+1} - \hat{I}_t\right) + \hat{S}^{inv}_t \tag{D5}$$

$$\widehat{ret}_{k,t} = \frac{1-\Delta}{1-\Delta+\bar{r}_k}E_t\hat{q}_{t+1} - \hat{q}_t + \frac{\bar{r}_k}{1-\Delta+\bar{r}_k}E_t\hat{r}_{k,t+1} \tag{D6}$$

$$\hat{r}_{k,t} = \sigma_a\hat{u}_t \tag{D7}$$

$$\begin{aligned} 0 \;=\; & \left(\frac{\bar{K}}{\bar{n}} - 1\right)\left(\widehat{ret}^{m1}_{k,t} - \hat{r}^{m1}_t\right) - \left(\hat{q}_{t-1} + \hat{K}_{t-1} - \hat{n}_{t-1}\right) \\ & + \left(\frac{\bar{K}}{\bar{n}} - 1\right)\frac{\bar{\Gamma}^\omega - \bar{\xi}\bar{G}^\omega}{\bar{\Gamma} - \bar{\xi}\bar{G}}\overline{\omega}\widehat{\bar{\omega}}_t + \left(\frac{\bar{K}}{\bar{n}} - 1\right)\frac{\bar{\Gamma}^\sigma - \bar{\xi}\bar{G}^\sigma}{\bar{\Gamma} - \bar{\xi}\bar{G}}\bar{\sigma}\hat{\sigma}_t \end{aligned} \tag{D8}$$

$$\begin{aligned} 0 \;=\; & \bar{\tilde{\lambda}}\left(\widehat{ret}_{k,t} - \hat{r}_t\right) - \left(1-\bar{\Gamma}\right)\frac{\bar{r}^k}{\bar{r}}\left(\frac{\bar{\Gamma}^{\omega\omega} - \bar{\tilde{\lambda}}\left(\bar{\Gamma}^{\omega\omega} - \bar{\xi}\bar{G}^{\omega\omega}\right)}{\bar{\Gamma}^\omega}\right)\overline{\omega}E_t\widehat{\bar{\omega}}_{t+1} \\ & + \frac{\bar{r}^k}{\bar{r}}\left[-\Gamma^\sigma + \bar{\tilde{\lambda}}\left(\bar{\Gamma}^\sigma - \bar{\xi}\bar{G}^\sigma\right) - \left(1-\bar{\Gamma}\right)\left(\frac{\bar{\Gamma}^{\omega\sigma} - \bar{\tilde{\lambda}}\left(\bar{\Gamma}^{\omega\sigma} - \bar{\xi}\bar{G}^{\omega\sigma}\right)}{\bar{\Gamma}^\omega}\right)\right]\bar{\sigma}E_t\hat{\sigma}_{t+1} \end{aligned} \tag{D9}$$

$$\begin{aligned} & \bar{n}\left(1+d\right)\left(\hat{n}_t + \hat{g}_t\right) \\ = \; & \frac{\bar{r}}{\bar{g}}\left(\bar{n} - \bar{K}\right)\hat{r}^{m1}_t + \frac{\bar{r}}{\bar{g}}\bar{n}\hat{n}_{t-1} + \frac{\bar{K}}{\bar{g}}\left(\overline{ret}_k\left(1-\bar{\xi}\bar{G}\right) - \bar{r}\right)\left(\hat{q}_{t-1} + \hat{K}_{t-1}\right) \\ & + \frac{\bar{K}}{\bar{g}}\overline{ret}_k\left(1-\bar{\xi}\bar{G}\right)\left(\widehat{ret}^{m1}_{k,t}\right) + \frac{\bar{K}}{\bar{g}}\overline{ret}_k\bar{\xi}\left(\bar{G}^\omega\overline{\omega}\widehat{\bar{\omega}}_t + \bar{G}^\sigma\bar{\sigma}\hat{\sigma}_t\right) \end{aligned} \tag{D10}$$

$$\widehat{ret}^{m1}_{k,t} = \widehat{ret}_{k,t-1} \tag{D11}$$

$$\hat{r}^{m1}_t = \hat{r}_{t-1} \tag{D12}$$

$$\hat{\psi}_t^w = \delta_w \hat{\psi}_{t-1}^w + (1-\delta_w)\hat{v}_{t-1}^w - \hat{\varepsilon}_t^{\pi^*} \tag{D13}$$

$$E_t \hat{v}_{t+1}^w = \hat{v}_t^w + \frac{(1-\delta_w\beta)^2}{(\delta_w\beta)^2}\frac{\delta_w}{1-\delta_w}\hat{\psi}_t^w - \frac{(1-\delta_w\beta)^2}{(\delta_w\beta)^2}\frac{\delta_w}{1-\delta_w}\hat{\pi}_t^w + \frac{(1-\delta_w\beta)^2}{(\delta_w\beta)^2}\frac{\widehat{mrs}_t - \hat{w}_t + \hat{\mu}_t^w}{(1+\phi_w\bar{\mu}_w\bar{\sigma}_w)} \tag{D14}$$

$$E_t \hat{\pi}_{t+1}^w = \hat{\pi}_t^w \left(\frac{2}{\beta} - \delta_w\right) + \hat{v}_t^w \left((1-\delta_w)(1+\delta_w)\right) \tag{D15}$$

$$+\hat{\psi}_t^w \left(\delta_w(1+\delta_w) - \frac{2}{\beta}\right) - \frac{2(1-\delta_w)(1-\delta_w\beta)}{(\delta_w\beta)}\frac{(\widehat{mrs}_t - \hat{w}_t + \hat{\mu}_t^w)}{(1+\phi_w\bar{\mu}_w\bar{\sigma}_w)}$$

$$\hat{w}_t = \hat{w}_{t-1} - \hat{g}_t + \hat{\pi}_t^w - \hat{\pi}_t^p \tag{D16}$$

$$\widehat{mrs}_t = \frac{1}{\eta}\hat{L}_t - \frac{\overline{\sigma^w}}{\eta}\frac{\delta_w}{1-\delta_w}\hat{\pi}_t^w + \frac{\overline{\sigma^w}}{\eta}\frac{\delta_w}{1-\delta_w}\hat{\psi}_t^w + \hat{S}_t^L - \hat{\lambda}_t \tag{D17}$$

12.3 Supply Block

$$\hat{\psi}_t^p = \delta_p \hat{\psi}_{t-1}^p + (1-\delta_p)\hat{v}_{t-1}^p - \hat{\varepsilon}_t^{\pi^*} \tag{S1}$$

$$E_t \hat{v}_{t+1}^p = \hat{v}_t^p + \frac{(1-\delta_p\beta)^2}{(\delta_p\beta)^2}\frac{\delta_p}{1-\delta_p}\hat{\psi}_t^p - \frac{(1-\delta_p\beta)^2}{(\delta_p\beta)^2}\frac{\delta_p}{1-\delta_p}\hat{\pi}_t^p + \frac{(1-\delta_p\beta)^2}{(\delta_p\beta)^2}\frac{\widehat{mc}_t + \hat{\mu}_t^p}{(1+\phi_p\bar{\mu}_p\bar{\sigma}_p)} \tag{S2}$$

$$E_t \hat{\pi}_{t+1}^p = \hat{\pi}_t^p \left(\frac{2}{\beta} - \delta_p\right) + \hat{v}_t^p \left((1-\delta_p)(1+\delta_p)\right) \tag{S3}$$

$$+\hat{\psi}_t^p \left(\delta_p(1+\delta_p) - \frac{2}{\beta}\right) - \frac{2(1-\delta_p)(1-\delta_p\beta)}{(\delta_p\beta)(1+\phi_p\bar{\mu}_p\bar{\sigma}_p)}(\widehat{mc}_t + \hat{\mu}_t^p)$$

$$\widehat{mc}_t = (1-\alpha)\hat{w}_t + \alpha\hat{r}_t^k \tag{S4}$$

$$\hat{L}_t = \widehat{mc}_t - \hat{w}_t + \hat{Y}_t \tag{S5}$$

$$\hat{k}_t = \widehat{mc}_t - \hat{r}_t^k + \hat{Y}_t \tag{S6}$$

12.4 Aggregate Relationships

$$\hat{\imath}_t = \theta^{int}\left(\hat{\imath}_{t-1} - \hat{\varepsilon}_t^{\pi^*}\right) + (1 - \theta^{int} + \theta^{\pi})E_t\left(\frac{\hat{\pi}_{t+3}^p + \hat{\pi}_{t+2}^p + \hat{\pi}_{t+1}^p + \hat{\pi}_t^p}{4}\right) + \theta^y E_t\left(\frac{\hat{Y}_{t+3} - \hat{Y}_{t-1}}{4}\right) + \frac{\hat{S}_t^{int}}{4} \quad \text{(A1)}$$

$$\hat{r}_t = \hat{\imath}_t - E_t\hat{\pi}_{t+1}^p \quad \text{(A2)}$$

$$\bar{Y}\hat{Y}_t = \bar{C}\hat{C}_t + \bar{I}\hat{I}_t + \overline{GOV}\hat{S}_t^{gov} \quad \text{(A3)}$$

$$\hat{k}_t = \hat{K}_{t-1} + \hat{u}_t - \hat{g}_t \quad \text{(A4)}$$

12.5 Shocks

12.5.1 Unit Roots

$$\hat{g}_t = \hat{g}_t^{gr} + \hat{g}_t^{iid} \tag{U1}$$

$$\hat{g}_t^{gr} = \rho_g \hat{g}_{t-1}^{gr} + \hat{\varepsilon}_t^{gr} \tag{U2}$$

$$\hat{g}_t^{iid} = \hat{\varepsilon}_t^{iid} \tag{U3}$$

$$\hat{\pi}_t^* = \hat{\pi}_{t-1}^* + \hat{\varepsilon}_t^{\pi^*} \tag{U4}$$

12.5.2 Stationary

$$\hat{S}_t^c = \rho^c \hat{S}_{t-1}^c + \hat{\varepsilon}_t^c \tag{X1}$$

$$\hat{S}_t^L = \rho^L \hat{S}_{t-1}^L + \hat{\varepsilon}_t^L \tag{X2}$$

$$\hat{S}_t^{inv} = \rho^{inv} \hat{S}_{t-1}^{inv} + \hat{\varepsilon}_t^{inv} \tag{X3}$$

$$\hat{S}_t^{gov} = \rho^{gov} \hat{S}_{t-1}^{gov} + \hat{\varepsilon}_t^{gov} \tag{X4}$$

$$\hat{S}_t^{int} = \rho^{int} \hat{S}_{t-1}^{int} + \hat{\varepsilon}_t^{int} \tag{X5}$$

$$\hat{\sigma}_t = \rho^{\sigma} \hat{\sigma}_{t-1} + \hat{\varepsilon}_t^{\sigma} \tag{X6}$$

$$\hat{\xi}_t = \rho^{\xi} \hat{\xi}_{t-1} + \hat{\varepsilon}_t^{\xi} \tag{X7}$$

$$\hat{\mu}_t^p = \hat{\varepsilon}^{\mu^p} \tag{X8}$$

$$\hat{\mu}_t^w = \hat{\varepsilon}^{\mu^w} \tag{X9}$$

13 REPORTING VARIABLES

- Remember that the ˆ-variables are the ones that come from the computer code.

13.1 Real Variables - Growth Rates

- Real GDP Growth:

$$DOT\_GDP_t = \ln Y_t - \ln Y_{t-1} = \ln \check{Y}_t - \ln \check{Y}_{t-1} + \ln g_t = \hat{Y}_t - \hat{Y}_{t-1} + \hat{g}_t + \ln(\bar{g})$$

- Real Consumption Growth:

$$DOT\_C_t = \ln C_t - \ln C_{t-1} = \ln \check{C}_t - \ln \check{C}_{t-1} + \ln g_t = \hat{C}_t - \hat{C}_{t-1} + \hat{g}_t + \ln(\bar{g})$$

- Real Investment Growth:

$$DOT\_INV_t = \ln I_t - \ln I_{t-1} = \ln \check{I}_t - \ln \check{I}_{t-1} + \ln g_t = \hat{I}_t - \hat{I}_{t-1} + \hat{g}_t + \ln(\bar{g})$$

- Real Wage Growth:

$$DOT\_W_t = \ln w_t - \ln w_{t-1} = \ln \check{w}_t - \ln \check{w}_{t-1} + \ln g_t = \hat{w}_t - \hat{w}_{t-1} + \hat{g}_t + \ln(\bar{g})$$

13.2 Nominal Variables - Growth Rates

- Inflation Growth:

$$DOT\_PAI_t = \ln(\pi^p_t) - \ln(\pi^p_{t-1}) = \ln(\check{\pi}^p_t) - \ln(\check{\pi}^p_{t-1}) + \ln\left(\varepsilon^{\pi^*}_t\right) = \hat{\pi}^p_t - \hat{\pi}^p_{t-1} + \hat{\varepsilon}^{\pi^*}_t$$

- Nominal Interest Rate Growth:

$$DOT\_INT_t = \ln(i_t) - \ln(i_{t-1}) = \ln(\check{i}_t) - \ln(\check{i}_{t-1}) + \ln\left(\varepsilon^{\pi^*}_t\right) = \hat{\imath}_t - \hat{\imath}_{t-1} + \hat{\varepsilon}^{\pi^*}_t$$

13.3 Real Variables - Stationary

- Labor Supply:

$$NODOT\_L_t = \ln(L_t) - \ln(\bar{L}) = \hat{L}$$